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LARGE CARDINALS AND THEIR EFFECT ON THE CONTINUUM FUNCTION ON REGULAR CARDINALS

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ABSTRACT
In this survey paper, we will summarise some of the more and less known results on the generalisation of the Easton theorem in the context of large cardinals. In particular, we will consider inaccessible, Mahlo, weakly compact, Ramsey, measurable, strong, Woodin, and supercompact cardinals. The paper concludes with a result from the opposite end of the spectrum: namely, how to kill all large cardinals in the universe.
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1. Introduction

One of the questions which stood at the birth of set theory as a mathematical discipline concerns the size of real numbers $\mathbb{R}$. Cantor conjectured that there is no subset of the real line whose cardinality is strictly between the size of the set of natural numbers and the size of all real numbers. With the axiom of choice, this is equivalent to saying that the size of $\mathbb{R}$ is the least possible in the aleph hierarchy:

$$\text{The Continuum Hypothesis, CH: } |\mathbb{R}| = 2^{\aleph_0} = \aleph_1.$$

Hilbert included this problem in 1900 as the number one question on his influential list of 23 open problems in mathematics.

It is well known now that CH is independent of the axioms of ZFC.[1] First Gödel showed in 1930s that CH is consistent with ZFC (using the constructible universe L), and then in 1960s Cohen showed that ¬CH is consistent with ZFC (with forcing). Regarding Cohen's result, one naturally inquires how much CH can fail in Cohen's model; it is a witness to the remarkable utility of the method of forcing that virtually the same proof gives the greatest possible variety of results: in principle,

(*) if κ is any cardinal with uncountable cofinality, then $2^{\aleph_0} = \kappa$ is consistent.

There is a small issue how to express (*) properly. We can view (*) as a statement about consistency of a theory, in which case κ should either be a parameter or should be definable in ZFC,[2] or (*) can be taken as a statement about pairs of models of ZFC. It is the latter approach which is more useful and general:

1 If ZFC is consistent, which we will assume throughout the paper.
2 E.g. $\aleph_{\omega+3}$, $\aleph_{\omega_1}$, or the first weakly inaccessible cardinal (if there is one).

Theorem 1.1 (Cohen, Solovay). *Let κ be a cardinal with uncountable cofinality in V, and assume $\kappa^\omega = \kappa$ in V. Then there is a cofinality-preserving extension $V[G]$ of V such that $V[G] \models (2^{\aleph_0} = \kappa)$.*

Easton [9] generalised this result to all regular cardinals. Let us write Card for the class of cardinals and Reg for the regular cardinals. Let F be a function from Reg to Card. Assume further that F satisfies for all κ, λ in Reg:

(i) $\kappa < \lambda \rightarrow F(\kappa) \leq F(\lambda)$.

(ii) $\kappa < \mathrm{cf}(F(\kappa))$.

Let us call such an F an *Easton function*. We say that an Easton function F is realised in a model M if $\mathrm{Reg} = \mathrm{Reg}^M$ and for all regular κ in M, $F(\kappa) = 2^\kappa$.

Theorem 1.2 (Easton). *Assume V satisfies* GCH *and let F be an Easton function definable over V. Then there is a definable cofinality-preserving proper-class forcing notion P such that if G is P-generic, then in $V[G]$,*

$$(\forall \kappa \in \mathrm{Reg})(2^\kappa = F(\kappa)),$$

i.e. F is realised in $V[G]$.

There are more general statements of Easton's theorem which remove the restriction of definability of F. Such generalisations usually require additional assumptions above ZFC: one can for instance start with an inaccessible cardinal κ and GCH below κ, and set $M = H(\kappa)$. Then M is a transitive model of ZFC + GCH. An Easton function F for M is now an element of $H(\kappa^+)$, and may not be definable over M. Easton's theorem now generalizes as follows:[3]

Theorem 1.3 (Easton, generalised version). *Let κ be an inaccessible cardinal and denote $M = V_\kappa$, and let F be an Easton function defined on regular cardinals $\alpha < \kappa$. Assume further that* GCH *holds below κ. Then there is a cofinality-preserving forcing notion of size κ such that if G is P-generic over V, then in $M[G]$,*[4]

$$(\forall \alpha \in \mathrm{Reg})(2^\alpha = F(\alpha)),$$

i.e. F is realised in $M[G]$.

Easton's theorem solves the problem of the possible behaviours of the continuum function on regular cardinals in models of ZFC in full generality. Mathematicians briefly conjectured that Easton's theorem could be extended to all cardinals – including the singular cardinals. However, Silver soon proved the following limiting theorem which shows that ZFC controls the continuum function more tightly on singular cardinals:

Theorem 1.4 (Silver). *Let κ be a singular strong limit cardinal of uncountable cofinality. If the set $\{\mu < \kappa \mid 2^\mu = \mu^+\}$ is stationary in κ, then $2^\kappa = \kappa^+$.*

SCH, Singular Cardinal Hypothesis, is a weakening of GCH and says that if κ is a singular strong limit cardinal, then $2^\kappa = \kappa^+$.[5] Silver's theorem claims that the validity of SCH at a

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[3] In the rest of the paper, we will not distinguish between these two versions of Easton's theorem.

[4] $M[G]$ is now viewed as a constructible closure of M relative to an additional predicate G.

[5] There are more versions of SCH, some of them formulated for all singular cardinals.

singular strong limit κ is determined by the continuum function on singular strong limit cardinals below κ: in particular, if SCH holds below κ, it must hold at κ.

Surprisingly, similar restrictions hold for regular cardinals which exhibit some combinatorial properties associated to large cardinals (see for instance Lemma 1.17), provided we wish to preserve these properties while realising an Easton function. Acknowledging the importance of large cardinals in current set theory, do we have a satisfactory analogue of Easton's theorem for extensions of ZFC with large cardinals? We will study this question in the following sections, defining all necessary notions as we proceed.

Remark 1.5. Due to lack of space, we completely disregard in this paper other possible, and interesting, generalisations of the Easton theorem: (i) one can for instance study the effect of former large cardinals on the continuum function (e.g. a regular κ with the tree property), (ii) consider other cardinal invariants in addition to 2^κ (see [6]), and finally (iii) consider the continuum function on all cardinals. Regarding (iii), as we mentioned above, there are some analogies between the restrictions valid for singular strong limit cardinals of uncountable cofinality (Silver's theorem) and restrictions valid for e.g. measurable cardinals (Lemma 1.17). However, there are also subtle differences which prevent an easy transfer of the respective results. In particular, in Lemma 1.17, the set A is required to be in a normal measure, not just stationary, as in Silver's theorem.

1.1 Large cardinals

We review some of the more basic large cardinals. The cardinals are listed in the increasing order of strength: inaccessible < Mahlo < weakly compact < Ramsey < measurable < strong < strongly compact, supercompact.[6] Slightly apart, there is the Woodin cardinal which in terms of consistency strength is roughly on the level of a strong cardinal, while it may not be even weakly compact (it is always Mahlo, though).

Proofs of results stated below as facts or mentioned in passing can be found in [14] or [15].

Definition 1.6. *Let κ be a regular uncountable cardinal. We say that κ is inaccessible if $2^\lambda < \kappa$ for every $\lambda < \kappa$ (this property is called being a strong-limit cardinal).*

Note that if GCH holds, then κ is inaccessible if and only if κ is regular and limit cardinal.

A slight strengthening of inaccessibility is Mahloness.

Definition 1.7. *We say that an inaccessible cardinal κ is Mahlo if the set of regular cardinals below κ is stationary.*

Lemma 1.8. *If κ is Mahlo, then the set of inaccessible cardinals is stationary below κ.*

6 < in this case means both the consistency strength and the provable implication: thus for instance a Mahlo cardinal has a strictly larger consistency strength than an inaccessible cardinal, and every Mahlo cardinal is an inaccessible cardinal. It is conjectured that the supercompact and strongly compact cardinals have the same consistency strength; in terms of the implication, a supercompact cardinal is always strongly compact, but not conversely.

Proof. Recall the definition of the function $\beth$: $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and $\beth_\gamma = \sup\{\beth_\delta \mid \delta < \gamma\}$ for γ limit. By the inaccessibility of κ, the set

$$A = \{\mu < \kappa \mid \beth_\mu = \mu\}$$

is a closed unbounded set of limit cardinals.

We want to show that every closed unbounded set $C \subseteq \kappa$ contains an inaccessible cardinals. By the previous paragraph, $C \cap A$ is a closed unbounded set. By Mahloness, the set of regular cardinals is stationary, and therefore it must meet $C \cap A$. Hence, there is $\mu \in C \cap A$ which is a regular cardinal. By the definition of A, μ is strong-limit and therefore inaccessible. □

As the next large cardinal after Mahlo cardinal, we review the weakly compact cardinal. There are many equivalent definitions of weak-compactness. The one we give first is formulated in terms of trees:

Definition 1.9. *An inaccessible κ is weakly compact if every κ-tree*[7] *has a cofinal branch.*

Note that this definition points to the original motivation for this cardinal: recall that König's theorem (that every ω-tree has a cofinal branch) can be used to prove the compactness theorem for the first-order logic. For a stronger logic which allows infinite quantifications, conjunctions and disjunctions, the similar proof goes through if κ is weakly compact (because the generalisation of König's theorem holds for κ).

An equivalent definition directly postulates a reflection property. We say that a formula φ in the language of set theory with two types of variables is Π^1_1 if it contains at the beginning a block of universal quantifiers over subsets of the target domain (second-order variables), followed by the usual first-order quantification over elements of the target domain (first-order variables). Thus $\forall X \exists x (x \in X)$ is true over a structure $(M, \in)$ if for every $A \subseteq M$ there is some $a \in M$ such that $a \in A$. We write $\varphi(R)$ to indicate that φ contains a free second-order variable R (we call R a parameter).

Fact 1.10. *The following are equivalent:*

(i) *κ is weakly compact.*

(ii) *κ is inaccessible and for every $R \subseteq V_\kappa$ and every Π^1_1 formula $\varphi(R)$,*

(1.1) *If* $(V_\kappa, \in, R) \models \varphi(R)$, *then*

$$(\exists \alpha < \kappa, \alpha \textit{ inaccessible})(V_\alpha, \in, V_\alpha \cap R) \models \varphi(R \cap V_\alpha).$$

Note that we can also view $(V_\kappa, \in, R)$ as a first-order structure with a predicate R; if κ is Mahlo, then the usual Löwenheim-Skolem theorem implies (ii) of Fact 2.10 for all *first-order formulas* $\varphi(R)$. However, to get (ii) for Π^1_1 formulas, the usual Löwenheim-Skolem theorem no longer suffices because now it should be applied over the first-order structure $(V_{\kappa+1}, V_\kappa, \in, R)$, and there is no guarantee it will yield a substructure of the form $(V_{\alpha+1}, V_\alpha, \in, R \cap V_\alpha)$.

Lemma 1.11. *Suppose κ is weakly-compact and x is a cofinal subset of κ. If $x \cap \alpha \in L$ for every $\alpha < \kappa$, then $x \in L$.*

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7 A tree of height κ whose levels have size $< \kappa$.

Proof. Sketch. Suppose $x \notin L$. Then there is a Π^1_1 sentence φ such that $(V_\kappa, \in, x) \models \varphi(x)$ if and only if x is not in L. φ contains a second-order quantifier which ranges over all subsets of κ which code levels of L of size at most κ and says that in no such level of L, x is constructed.

By weak-compactness, φ is reflected to some $\alpha < \kappa$, which gives $(V_\alpha, \in, x \cap \alpha) \models \varphi(x \cap \alpha)$, which is equivalent to $x \cap \alpha \notin L$, contradicting our initial assumption. □

A weakly compact cardinal has another useful characterisation by means of colourings. If κ is a regular cardinal, then a colouring of two-element subsets of κ by two colours is a function $f : [\kappa]^2 \to 2$. We say that $H \subseteq \kappa$ is homogeneous for f if $f \restriction [H]^2$ has size 1.

Fact 1.12. *The following are equivalent for an inaccessible κ:*
(i) κ is weakly compact.
(ii) Every colouring $f : [\kappa]^2 \to 2$ has a homogeneous set of size κ.

By considering more complex colourings, we can obtain a stronger large cardinal notion:

Definition 1.13. *Let $\kappa > \omega$ be an inaccessible cardinal. We say that κ is a Ramsey cardinal if every colouring $f : [\kappa]^{<\omega} \to 2$ has a homogeneous set of size κ.*

By definition, every Ramsey cardinal is weakly compact. Moreover, one can show that if there is a Ramsey cardinal, then $V \neq L$. Thus being Ramsey is a substantial strengthening of weak compactness which is compatible with L.

Another cardinal we will mention is the measurable cardinal:

Definition 1.14. *We say that an inaccessible κ is measurable if there is a non-principal[8] κ-complete[9] ultrafilter U on κ. U is often called a measure.*

Fact 1.15. *The following are equivalent:*
(i) κ is measurable.
(ii) There is an elementary embedding[10] $j : V \to M$, where M is a transitive class, $j \restriction \kappa = id$ and $j(\kappa) > \kappa$. (We call κ the critical point of j.)
If (ii) holds, we can find an embedding $j' : V \to M'$ which in addition satisfies that $\kappa^+ = (\kappa^+)^{M'}$, $H(\kappa^+)^{M'} = H(\kappa^+)$, and M' is closed under κ-sequences in V.

We should say something about proving (i)→(ii) because it features the important concept of an *ultrapower*. Assume that U is a measure on κ. For $f, g : \kappa \to V$ define $f \equiv g \Leftrightarrow \{\xi < \kappa \mid f(\xi) = g(\xi)\} \in U$. For every $f : \kappa \to V$, define

$$[f] = \{g \mid g : \kappa \to V \;\&\; f \equiv g\}.$$

We would like to say that the collection of all $[f]$'s forms a partition of the class of all functions $\kappa \to V$; this is the case, but it presents the problem that this collection is a class

8 For no $\alpha < \kappa$, $\{\alpha\} \in U$.

9 If X_i, $i < \mu < \kappa$ are in U, then $\bigcap_{i<\mu} X_i$ is in U.

10 j is a proper class; thus we should view this definition as taking place in GB set theory, or more technically – but preferably – as a statement expressible in ZFC because the relevant part of j which we need, $j \restriction H(\kappa^+)$, is a set.

of classes, making it an illegal object in set theory. We will therefore identify $[f]$ with the sets in $[f]$ of minimal rank. Using this identification, denote

$$\mathrm{Ult}(V, U) = \{[f] \mid f : \kappa \to V\}.$$

Define the interpretation of $\in$ on elements of $\mathrm{Ult}(V, U)$: $[f] \in [g] \Leftrightarrow \{\xi < \kappa \mid f(\xi) \in g(\xi)\} \in U$.

Theorem 1.16 (Łos)**.** *For every φ and $f_1, \dots, f_n$:*

$$\mathrm{Ult}(V, U) \models \varphi[[f_1], \dots, [f_n]] \Leftrightarrow \{\xi < \kappa \mid \varphi(f_1(\xi), \dots, f_n(\xi))\} \in U. \tag{1.2}$$

By ω_1-completeness of the measure U, the relation $\in$ on $\mathrm{Ult}(V, U)$ is well-founded, and one can therefore collapse the structure $(\mathrm{Ult}(V, U), \in)$, obtaining a transitive proper class model. The proof (i)→(ii) is finished by taking for j the composition of the canonical ultrapower embedding $j' : V \to \mathrm{Ult}(V, U)$ defined by

$$j'(x) = [c_x],$$

where $c_x : \kappa \to \{x\}$, and of the collapsing isomorphism π:

$$j = \pi \circ j'.$$

We say that U is *normal* if

$$[id] = \kappa. \tag{1.3}$$

One can show that if κ is measurable, there always exists a normal measure. Property (1.3) is useful for computing information about ultrapowers; see Lemma 1.17 for an application.

Lemma 1.17. *Assume κ is measurable and let U be a normal measure. If $A = \{\alpha < \kappa \mid 2^\alpha = \alpha^+\}$ is in U, then $2^\kappa = \kappa^+$.*

Proof. Let $\mathrm{Ult}(V, U)$ be the transitive collapse of the ultrapower, as discussed above after Fact 1.15. By Łos theorem, $A \in U$ implies

$$\mathrm{Ult}(V, U) \models 2^{[id]} = [id]^+$$

which is by normality the same as

$$\mathrm{Ult}(V, U) \models 2^\kappa = \kappa^+.$$

As stated in Fact 1.15, $\kappa^+ = (\kappa^+)^{\mathrm{Ult}(V,U)}$, and $H(\kappa^+) = (H(\kappa^+))^{\mathrm{Ult}(V,U)}$. This implies $\mathcal{P}(\kappa) = (\mathcal{P}(\kappa))^{\mathrm{Ult}(V,U)}$. Therefore any bijection $g \in \mathrm{Ult}(V, U)$ between $(\kappa^+)^{\mathrm{Ult}(V,U)}$ and $\mathcal{P}(\kappa)^{\mathrm{Ult}(V,U)}$ is a bijection between κ^+ and $\mathcal{P}(\kappa)$ in V, proving $2^\kappa = \kappa^+$. □

A useful set which belongs to any normal measure is

$$I = \{\alpha < \kappa \mid \alpha \text{ is inaccessible}\}.$$

I is stationary and co-stationary, i.e. $(\kappa \setminus I)$ is also stationary. I is in every normal measure because $\kappa = [id]$ is inaccessible in $\mathrm{Ult}(V, U)$; by Łos theorem this implies that I is in U. By a similar argument one can show that if C is club in κ, then $C \in U$: in the ultrapower, $\kappa \in j(C)$, which by Łos theorem is equal to $C \in U$. Note that Lemma 1.17 depends on ultrafilter U in the following sense. Denote

$$A = \{\alpha < \kappa \mid 2^\alpha = \alpha^+\}.$$

To argue that $2^\kappa = \kappa^+$ it suffices to find *at least one* normal measure U which contains A. As we discussed, if A is club or a set of inaccessibles, then all normal measures contain A. However, if A is just stationary, then it is not the case in general that there is some normal measure U which contains A. In fact, it is consistent that A is stationary and $2^\kappa > \kappa^+$ (see Lemma 2.14).

By strengthening the properties of the elementary embedding in the definition of a measurable cardinal, we get the notion of a strong cardinal. For more motivation and properties of strong cardinals, see Section 2.3.

Definition 1.18. *We say that an inaccessible cardinal κ is $H(\lambda)$-strong, $\kappa < \lambda$ regular, if there is an elementary embedding $j : V \to M$ with critical point κ, $j(\kappa) > \lambda$, $H(\lambda) \subseteq M$, and M is closed under κ-sequences in V.*

We say that κ is strong if it is $H(\lambda)$-strong for every regular $\lambda > \kappa$.

By definition, being measurable is the same as being $H(\kappa^+)$-strong.

By strengthening the closure properties of the target model M in the definition of a strong cardinal, we obtain an even stronger notion of a supercompact cardinal (see Definition 1.21). However, we first define the notion of a strongly compact cardinal, using a generalisation of the ultrafilter definition of a measurable cardinal. In preparation for the definition, let us define the following: Let $\kappa \leq \lambda$ be cardinals, κ regular, and set

$$P_\kappa\lambda = \{x \subseteq \lambda \mid |x| < \kappa\}.$$

For $x \in P_\kappa\lambda$, define

$$\hat{x} = \{y \in P_\kappa\lambda \mid x \subseteq y\}.$$

Finally, define

$$F(\kappa, \lambda) = \{X \subseteq P_\kappa\lambda \mid (\exists x \in P_\kappa\lambda)\ \hat{x} \subseteq X\}.$$

We call $F(\kappa, \lambda)$ *a fine filter on $P_\kappa\lambda$.*

Lemma 1.19. *$F = F(\kappa, \lambda)$ is a κ-complete filter.*

Proof. Follows because for $\{x_i \mid i < \mu < \kappa\} \subseteq P_\kappa\lambda$,

$$\bigcap_{i<\mu} \hat{x}_i = \widehat{\bigcup_{i<\mu} x_i}.$$

□

Definition 1.20. *Assume $\kappa \leq \lambda$ are cardinals, κ inaccessible. We call κ λ-strongly compact if the fine filter $F(\kappa, \lambda)$ can be extended into a κ-complete ultrafilter on $P_\kappa\lambda$. We call κ strongly compact if it is λ-strongly compact for all $\lambda \geq \kappa$.*

Strongly compact cardinals are much stronger than measurable cardinals (regarding consistency strength); however, by a result of Magidor from 70s the first measurable cardinal can be strongly compact.

By demanding that there is a κ-complete ultrafilter extending $F(\kappa, \lambda)$ which is also *normal* (we will not define this notion, see [14], p. 374), we get the notion of a *supercompact* cardinal. A characterisation of supercompactness by means of elementary embeddings is very convenient:

Definition 1.21. *Let κ be an inaccessible cardinal, and let $\lambda \geq \kappa$ be a cardinal. We say that κ is λ-supercompact if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$. A cardinal κ is supercompact if it is λ-supercompact for every $\lambda \geq \kappa$.*

Finally, we define a large cardinal notion due to Woodin which he used in the analysis of the Axiom of Determinacy.

Definition 1.22. *Let $\delta > \omega$ be an inaccessible cardinal. We say that δ is a Woodin cardinal if for every function $f : \delta \to \delta$ there is a $\kappa < \delta$ with $f''\kappa \subseteq \kappa$ and there is $j : V \to M$ with critical point κ such that $V_{j(f)(\kappa)} \subseteq M$.*

A Woodin cardinal is always Mahlo, but may not be weakly compact. Its consistency strength is quite high (by definition, there are many cardinals on the level of a $H(\mu)$-strong cardinal, for some μ, below a Woodin cardinal).

2. The continuum function with large cardinals

Assume κ is a large cardinal in V which satisfies GCH and F is an Easton function. Can we find a generic extension of V which realises F and preserves the largeness of a fixed large cardinal κ? Clearly, a necessary condition on F is that it should keep κ strong limit. We can formulate this property globally for the class of large cardinals we wish to preserve. Let Γ be a class of regular cardinals. We say that F *respects* Γ if

$$(\forall \kappa \in \Gamma)(\forall \mu \in \text{Reg} \cap \kappa)(F(\mu) < \kappa). \tag{2.4}$$

In anticipation of the generalisation of Easton's theorem to large cardinals, we can tentatively formulate two distinguishing criteria, (R^-), (R) and (L^-), (L), which help to characterise large cardinals according to their sensitivity to the manipulation with the continuum function:

(R^-) Cardinals without obvious reflection properties relevant to the continuum function[11] such as inaccessible, Mahlo, weakly compact, and Woodin or Ramsey cardinals.

(R) Cardinals with reflection properties relevant to the continuum function, such as measurable cardinals.

A typical effect of reflection of measurable cardinals regarding the continuum function is captured in Lemma 1.17 above.

Remark 2.1. The notion of reflection is often used in a broad sense (for instance, Fact 2.10 provides a notion of reflection for Π^1_1-formulas). In (R^-) and (R), we use it in a very restricted sense: κ has a reflection property (relevant to the continuum function) if 2^κ depends on the values of 2^α, for $\alpha < \mu$.

A different classification is based on what is called *fresh subsets*:

[11] Of course, only after we generalise Easton's theorem to these cardinals we know for certain that they have no "hidden" reflection properties.

Definition 2.2. *Let $M \subseteq N$ be two transitive models of set theory with the same ordinals. Let κ be a cardinal in N. We say that $x \subseteq \kappa$ is fresh if $x \in N \setminus M$ and for all $\alpha < \kappa$, $x \cap \alpha \in M$.*

For instance Cohen forcing[12] at a regular cardinal κ adds a fresh subset κ.

(L^-) Cardinals which are not obviously influenced by fresh subsets such as inaccessible and Mahlo cardinals.

(L) Cardinals which may be destroyed by adding fresh subsets such as weakly compact cardinals, or measurable cardinals.

Lemma 1.11 identifies this restriction for weakly compact cardinals.

As we will see, the first distinction (R^-) and (R) is relevant for the possible patterns of the continuum function which can be realised, while the second distinction (L^-) and (L) is relevant for the appropriate method of forcing.

The following forcing, defined in Easton [9], will be refereed to as the *product-style Easton forcing*, and denoted it P_F^{product}.

Definition 2.3. *Let F be an Easton function. For all regular cardinals α, define Q_α to be the Cohen forcing* $\mathrm{Add}(\alpha, F(\alpha))$. *Define*

$$P_F^{\text{product}} = \prod_{\alpha \in \mathrm{Reg}}^{\text{Easton}} Q_\alpha,$$

where the upper index indicates that the forcing has the "Easton support": for every inaccessible α and any condition $p \in P_F^{\text{product}}$, $\mathrm{dom}(p) \cap \alpha$ is bounded in α.

Note in particular that if there are no inaccessible cardinals, then the forcing is just a full-support product of Cohen forcings. It is relatively straightforward to compute that if GCH holds in the ground model, then P_F^{product} preserves all cofinalities and forces $2^\alpha = F(\alpha)$, for all regular α.

As we indicated above in the paragraph after the definition of a fresh subset, a product-style forcing will not be good enough for preservation of large cardinals with reflection as in Lemma 1.11. In anticipation of a solution to this problem, we define a variant of Easton forcing which appeared already in [17]. For this definition, let us first define some notions. If F is an Easton function, let C_F be the closed unbounded class of limit cardinals which are the closure points of F: i.e.

$$C_F = \{\alpha \mid \alpha \text{ limit cardinal } \ \& \ (\forall \beta \in \alpha \cap \mathrm{Reg})(F(\beta) < \alpha)\}.$$

Notice that if F respects Γ, see (2.4), then Γ is included in C_F.

Definition 2.4. *Let F be an Easton function. By reverse Easton forcing we mean the forcing P_F defined as follows. For every pair (α, β) of successive elements of C_F, let us write*

$$Q_{\alpha,\beta} = \prod_{\gamma \in [\alpha,\beta) \cap \mathrm{Reg}}^{\text{Easton}} \mathrm{Add}(\gamma, F(\gamma)).$$

[12] If α is a limit ordinal and $\beta > 0$ is an ordinal, we define the Cohen forcing at α for adding β-many subsets of α, $\mathrm{Add}(\alpha, \beta)$, as the collection of all functions from $\alpha \times \beta$ to 2 with domain of size $< |\alpha|$. Ordering is by reverse inclusion. Of course, $\mathrm{Add}(\alpha, \beta)$ is equivalent to $\mathrm{Add}(|\alpha|, |\beta|)$, but the more general notation is often useful.

P_F is the iteration $(\langle P_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle \dot{Q}_\alpha \mid \alpha \in \mathrm{Ord}\rangle)$ with Easton support such that $\dot{Q}_\alpha$ is the canonical name for the trivial forcing whenever α is not in C_F. If α is in C_F, let $\dot{Q}_\alpha$ be a name for the forcing $Q_{\alpha,\beta}$, where β is the successor of α in C_F.

2.1 Inaccessible and Mahlo cardinals

Let F be an Easton function respecting inaccessible cardinals, i.e. respecting $\Gamma = \{\alpha \mid \alpha$ is inaccessible$\}$ according to (2.4). To generalise Easton's theorem to F, it suffices to check that the forcing P_F^{product} preserves cofinalities of all $\kappa \in \Gamma$. As we indicated after Definition 2.3, cofinalities are preserved for all cardinals if V satisfies GCH, which yields the following theorem:

Theorem 2.5. *Let V satisfy* GCH *and let F be an Easton function respecting inaccessible cardinals. Let A_0 be the class of all inaccessible cardinals. Then in any generic extension $V[G]$ by P_F^{product}, the set of inaccessible cardinals coincides with A_0.*

One can formulate a version of the theorem for Mahlo cardinals.

Theorem 2.6. *Let V satisfy* GCH *and let F be an Easton function respecting Mahlo cardinals. Let A_0 be the class of all Mahlo cardinals. Then in any generic extension $V[G]$ by P_F^{product}, the set of Mahlo cardinals coincides with A_0.*

Proof. Let G be P_F^{product}-generic and let κ be a Mahlo cardinal in V. Since the set of inaccessible cardinals I is stationary in κ in V, $C_F \cap I$ is also stationary. It follows by Theorem 2.5 that all inaccessible $\alpha \in C_F \cap I$, and also κ, remain inaccessible in $V[G]$. To finish the argument, it suffices to check that $C_F \cap I$ is still stationary in $V[G]$. Factor P_F^{product} into $P_0 \times P_1$ such that P_1 is κ-closed and P_0 is κ-cc:[13] P_1 preserves stationary subsets of κ because it is κ-closed; as P_1 forces that P_0 is κ-cc, P_0 preserves stationary subsets over V^{P_1}. Thus $P = P_0 \times P_1$ preserves stationary subsets of κ, and in particular stationarity of $C_F \cap I$. □

Actually, the reverse Easton iteration P_F achieves the same result here. The point is that for every Mahlo κ, one can show that $(P_F)_\kappa$, the restriction of P_F to κ, is κ-cc, and the tail iteration is forced to be κ-closed.

Remark 2.7. We have argued that the relevant forcings do not kill inaccessible or Mahlo cardinals. To get the results above, we also need to argue that the forcings do not create new large cardinals. However, notice that P_F^{product} and P_F cannot create new inaccessible cardinals because these forcings preserve cofinalities, and therefore a non-inaccessible cardinal α in the ground model must remain non-inaccessible in the extension. Similarly, a non-stationary set of inaccessible cardinals cannot become stationary, and thus new Mahlo cardinals cannot be created.

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[13] P_0 is defined as P_F^{product}, but with the domain of the functions in the product limited to $\kappa \cap \mathrm{Reg}$; similarly, P_1 has the domain limited to $\mathrm{Reg} \setminus \kappa$.

It is easy to find an example where the product-style Easton forcing P_F^{product} destroys weak-compactness of some cardinal κ, over some well-chosen ground model such as L.

Lemma 2.8. *Assume that κ is weakly compact and let F be an Easton function. Then over L, P_F^{product} kills weak-compactness of κ.*

Proof. P_F^{product} factors at κ to $P_0 \times P_1 \times P_2$, where P_0 is P_F^{product} restricted to regular cardinals $< \kappa$, P_1 is the forcing $\mathrm{Add}(\kappa, F(\kappa))$, and P_2 is the restriction to regular cardinals $> \kappa$. We argue that P_1 kills the weak-compactness of κ, and neither P_0, nor P_2 can resurrect it.

The fact that P_1 kills weak-compactness of κ follows from Lemma 1.11 (because it adds many fresh subsets of κ over L). It follows that after forcing with P_1, there exists a κ-tree without a cofinal branch. Since P_2 cannot add a branch to a κ-tree because it is κ^+-distributive over V^{P_1}, κ is not weakly compact in $V^{P_1 \times P_2}$.

Finally notice that P_0 is κ-Knaster in $V^{P_1 \times P_2}$ by the usual Δ-lemma argument (and the fact that κ is Mahlo here). Using the fact that a κ-Knaster forcing cannot add a branch to a κ-tree (see [1]), we conclude that in V^{P_F} there exist a κ-tree without a cofinal branch, contradicting weak-compactness of κ. □

In order to formulate Theorem 2.6 for weakly compact cardinals, we need to introduce a very universal technique for verification of preservation of large cardinals. This technique uses the characterisation of many large cardinals by means of suitable elementary embeddings between transitive sets or classes. In order to show that a certain large cardinal κ remains large in a generic extension, it suffices to check that the original embedding from V "lifts" to an embedding in the generic extension (this is in general easier than to verify that there exists *an* elementary embedding in the extension). The following lemma of Silver is the key ingredient:

Lemma 2.9 (Silver). *Assume M and N are transitive models of* ZFC, *$P \in M$ is a forcing notion, and $j : M \to N$ is an elementary embedding. Let G be P-generic over M, and let H be $j(P)$-generic over N. Then the following are equivalent:*
(i) $(\forall p \in G)(j(p) \in H)$.
(ii) There exists an elementary embedding $j^+ : M[G] \to N[H]$ such that $j^+(G) = H$ and $j^+ \restriction M = j$.

We say that j^+ is a *lifting of j*. If j has some nice property (like being an extender embedding), the lifting j^+ will often have it as well. More details about these concepts can be found in [5].

This is a useful characterisation of weakly compact cardinals (proof can be found in [5]):

Fact 2.10. *Let κ be an inaccessible cardinal. The following are equivalent.*
(i) κ is weakly compact.
(ii) For every transitive set M with $|M| = \kappa$, $\kappa \in M$, and ${}^{<\kappa}M \subseteq M$, there is an elementary embedding $j : M \to N$ where N is transitive, $|N| = \kappa$, ${}^{<\kappa}N \subseteq N$, and the critical point of j is κ.

Now, using the characterisation of weak-compactness by elementary embeddings, one can show:

Theorem 2.11. *Let V satisfy* GCH *and let F be an Easton function respecting weakly compact cardinals. Let A_0 be the class of all weakly compact cardinals. Then in any generic extension $V[G]$ by P_F, the set of weakly compact cardinals coincides with A_0.*

Proof. The proof has two parts: Part 1 proves that all weakly compact cardinals in V remain weakly compact in $V[G]$. In Part 2, which corresponds to Remark 2.7 above, we argue that the forcing does not create new weakly compact cardinals.

Part 1.

The proof is given in [3]; we will only briefly identify the main points, assuming some familiarity with lifting arguments. The proof is similar to an argument in [5], section 16 – when one uses the forcing P_F – with one extra twist to be resolved: assuming κ is weakly compact, in [5], one forces below κ with a reverse Easton forcing which at every inaccessible $\alpha < \kappa$ forces with $\mathrm{Add}(\alpha, 1)$. At κ, one can force with $\mathrm{Add}(\kappa, \mu)$ for an arbitrary μ because any κ-tree which supposedly does not have a cofinal branch is captured by a subforcing of $\mathrm{Add}(\kappa, \mu)$ which is isomorphic to $\mathrm{Add}(\kappa, 1)$; thus the preparation below κ matches the forcing at κ, making it possible to use a standard lifting argument with a master condition. In Theorem 2.11, the preparation below κ is determined by F so it may not be possible to force just with $\mathrm{Add}(\alpha, 1)$ at every inaccessible $\alpha < \kappa$; in particular if $j : M \to N$ is an embedding ensured by Fact 2.10, we need to force with $\mathrm{Add}(\kappa, j(F)(\kappa))$ on the N-side; this introduces a mismatch between the forcings at κ between M and N: $\mathrm{Add}(\kappa, 1)$ vs. $\mathrm{Add}(\kappa, j(F)(\kappa))$. In order to lift to $j(\mathrm{Add}(\kappa, 1))$, one therefore needs to make sure to have on the N-side available the generic filter g for $\mathrm{Add}(\kappa, 1)$. In [3], the solution is to include g on the first coordinate of the generic filter for $\mathrm{Add}(\kappa, j(F)(\kappa))$. The rest of the argument is standard.

Part 2.

The situation of a weakly compact cardinal is a bit more complicated than in the analogous Remark 2.7. By Kunen's construction [16], it is possible to turn a weakly compact cardinal κ into a Mahlo non-weakly compact cardinal by forcing a κ-Souslin tree, and resurrect its weak-compactness by forcing with the Souslin tree added earlier. However, it is easy to check that this kind of anomaly will not occur with our forcing.

Let κ be a Mahlo non-weakly compact cardinal in V which is a closure point of F; it follows there is a κ-tree T in V which has no cofinal branch in V. Denote $R = (P_F)_\kappa$, and $\dot{Q} = \mathrm{Add}(\kappa, F(\kappa))$; it suffices to check that $R * \dot{Q}$ cannot add a branch through T. R cannot add a cofinal branch because it is κ-Knaster. Over V^R, $\dot{Q}$ cannot add a branch to T because it is κ-closed (if p in $\dot{Q}$ forced that $\dot{B}$ is a cofinal branch through T, then one could find a decreasing sequence of conditions $\langle p_i \mid i < \kappa \rangle$, $p_0 = p$ and a $\leq_T$-increasing sequence $\langle b_i \mid i < \kappa \rangle$ such that $p_i \Vdash b_i \in T$; the sequence $\langle b_i \mid i < \kappa \rangle$ would be a cofinal branch in T in V^R). □

Thus for inaccessible, Mahlo and weakly compact cardinals, there are no restrictions on the Easton functions F which can be realised, except that these cardinals must be closure points of F. In particular, the reflection property identified in Lemma 1.11 did have an effect on the technique (P_F over P_F^{product}), but not on the result. In the next section, we learn that the case of measurable cardinals is far more complicated.

It follows from Lemma 1.17 that to preserve measurable cardinals, we must expect that the full generalisation along the lines of Theorems 2.6 and 2.11 cannot be achieved. There are two easy properties to notice regarding restrictions on the continuum function by measurable cardinals:

(a) There is an obvious asymmetry in the sense that Lemma 1.17 prohibits 2^κ "jumping up" with respect to the values 2^α for $\alpha < \kappa$, while "jumping down" is perfectly possible. See Lemma 2.12.

(b) The restrictions which a measurable cardinal κ puts on the continuum function also depend on the normal measures which exist on κ (and not only on the fact that κ is measurable). See Lemma 2.14.

We first deal with (a).

Lemma 2.12. *Assume that κ is measurable and $2^\kappa > \kappa^+$. Let P be the collapsing forcing $\mathrm{Col}(\kappa^+, 2^\kappa)$ which collapses 2^κ to κ^+ by functions of size at most κ. Then in V^P, κ is still measurable and $2^\kappa = \kappa^+$.*

Proof. By κ^+-closure of P, every measure on κ in V remains a measure in V^P because P does not add new subsets of κ to measure (nor new κ-sequences of such sets). Notice that the result did not assume that $\{\alpha < \kappa \mid 2^\alpha = \alpha^+\}$ is big in the sense of some measure on κ. □

We will deal with (b) after we define the notion of an $H(\lambda)$-strong cardinal.

Apart from the easy observations (a) and (b), we in addition have:

(c) The consistency strength of a measurable cardinal κ with $2^\kappa > \kappa^+$ is $o(\kappa) = \kappa^{++}$, see [12]. Thus to play with the continuum function and preserve measurability of cardinals, one typically needs to assume that these cardinals are larger than measurable in the ground model.

In view of (c), we now define a suitable strengthening of measurability.

Definition 2.13. *We say that an inaccessible cardinal κ is $H(\lambda)$-strong, $\kappa < \lambda$ regular, if:*

(i) There is an elementary embedding $j : V \to M$ with critical point κ, $j(\kappa) > \lambda$, such that

(ii) $H(\lambda) \subseteq M$, and M is closed under κ-sequences in V.

We say that κ is strong if it is $H(\lambda)$-strong for every regular $\lambda > \kappa$.

We note that with GCH, κ being $H(\kappa^{++})$-strong is equivalent to having Mitchell order of $\kappa^{++}+1$, a slight strengthening of the assumption identified by [12] as optimal for obtaining the failure of GCH at a measurable cardinal.

As promised, we now deal with the property (b).

Lemma 2.14. *Assume κ is $H(\kappa^{++})$-strong and that* GCH *holds in the universe. Denote $I = \{\alpha < \kappa \mid \alpha \text{ is inaccessible}\}$.*

Then there exist a stationary subset X of I, distinct normal measures U, W on κ, and a forcing notion P such that:

(i) $X \in W$ and $(I \setminus X) \in U$,

(ii) Assume G is P-generic. In $V[G]$, κ is measurable, $2^\kappa = \kappa^{++}$, $2^\alpha = \alpha^+$ for all $\alpha \in X$, and $2^\alpha = \alpha^{++}$ for all $\alpha \in (I \setminus X)$.

In particular, W cannot be extended into a normal measure in V^P.

Proof. Let U, W be two distinct normal measures on κ in V. We know that I is in both U and W; therefore for some $A \subseteq I$, $A \in U$ and $B = (I \setminus A) \in W$ (if U and W agreed on all subsets of I, they would agree on all subsets of κ).

Let $j : V \to M$ be an elementary embedding witnessing $H(\kappa^{++})$-strength of κ. Without loss of generality assume that $\kappa \in j(A)$ (and $\kappa \notin j(B)$). We define P so that $B = X$ is as desired.

Let Q be the standard reverse Easton iteration which at every $\alpha \in (I \setminus X)$ forces with $\mathrm{Add}(\alpha, \alpha^{++})$. By an argument involving "surgery", see [5], one can show that there is a forcing $\dot{R}$ such that denoting $P = Q * \dot{R}$, in V^P all cofinalities are preserved, $2^\kappa = \kappa^{++}$, and κ is measurable. Moreover, in V^P, X and $(I \setminus X)$ are stationary subsets of inaccessible cardinals such that $2^\alpha = \alpha^{++}$ for $\alpha \in (I \setminus X)$, and $2^\alpha = \alpha^+$ for $\alpha \in X$.

It follows that U extends to a normal measure in V^P, while by Lemma 1.17, W (and any other normal measure containing X) cannot extend into a normal measure in V^P. □

This lemma should be understood as follows: while W prohibits certain values of the continuum function in V because $X \in W$ (e.g. implies $2^\kappa = \kappa^+$), this restriction is not persistent to larger models: in V^P, $2^\kappa = \kappa^{++}$ is possible even though X is still a stationary subset composed of inaccessible cardinals. This scenario is made possible by the assumption that there is at least one embedding j in V for which the set $I \setminus X$ is big – using this j we can kill all normal measures which contain X, while ensuring that some normal measures still exist in V^P.

These consideration lead to the following theorem (see [10]):

Theorem 2.15. *Let F be an Easton function respecting every κ which is $H(F(\kappa))$-strong, and assume* GCH *holds in the universe. There is a cofinality-preserving iteration P which realises F such that whenever G is P-generic over V, we have:*

Whenever in V, κ is $H(F(\kappa))$-strong and there is $j : V \to M$ witnessing $H(F(\kappa))$-strength of κ such that

$$j(F)(\kappa) \geq F(\kappa), \tag{2.5}$$

then κ remains measurable in $V[G]$.

The proof is beyond the scope of this paper, but let us at least comment on the method of proof. As we mentioned in Lemma 2.14, the manipulation of 2^κ with κ measurable using the Cohen forcing and Woodin's "surgery argument" requires us to use an extra forcing denoted $\dot{R}$ in the proof of Lemma 2.14. It seems quite hard to incorporate this extra forcing at every relevant stage into a global result along the lines of Theorem 2.15. Instead, to prove Theorem 2.15 we use the generalised product-style α-Sacks forcing $\mathrm{Sacks}(\alpha, \beta)$, for an inaccessible α and an ordinal $\beta > 0$ (see [10] for details): P is a reverse Easton iteration defined similarly as in Definition 2.4 with $\mathrm{Add}(\gamma, F(\gamma))$ replaced by $\mathrm{Sacks}(\gamma, F(\gamma))$ whenever γ is an inaccessible closure point of F.[14] The use of Sacks forcing has the advantage that to lift an embedding, no extra forcing $\dot{R}$ is required.

[14] Since one mixes the α-Sacks forcing with the α^+-Cohen forcing (and other Cohen forcings – but only the stage α^+ requires an argument), one needs to argue that they work well together: in particular, one can show (see [10]) that $\mathrm{Sacks}(\alpha, F(\alpha))$ forces that $\mathrm{Add}(\alpha^+, F(\alpha^+))$ is still α^+-distributive. In fact, this is true for any α^+-closed forcing in place of $\mathrm{Add}(\alpha^+, F(\alpha^+))$.

The property (2.5) is essential for lifting the embedding at κ, and captures the degree of reflection which F needs to satisfy for preservation of measurability of κ. The proof is relatively straightforward when $F(\kappa)$ is regular, but is more involved when $F(\kappa)$ is a singular cardinal (the most difficult case is when $F(\kappa)$ has cofinality $> \kappa^+$ in V and is singular in V, but is regular in M, where $j : V \to M$ is an embedding witnessing (2.5)).

Note that the apparent lack of uniformity in the statement of the theorem (the condition (2.5)) is unavoidable as illustrated in Lemma 2.14. Also note that the use of $H(F(\kappa))$-strong cardinals is almost optimal, as mentioned above in the discussion of property (c).

Remark 2.16. We have not checked whether every measurable cardinal κ in $V[G]$ is measurable also in V, obtaining an analogue of Theorems 2.6 and 2.11, but we consider it likely.

2.4 Ramsey, Woodin and supercompact cardinals

We shall more briefly review results for some other large cardinals, most notably Ramsey, Woodin and supercompact.

A Ramsey cardinal, see Definition 1.13, is large enough to imply $V \neq L$, but it may not be measurable (and its consistency strength is less than measurability). In the classification following (2.4), Ramsey cardinals are in (R$^-$) and (L). We will see below in Theorem 2.17 that indeed Ramsey cardinals have no reflection properties relevant for the continuum function.

In terms of consistency, Woodin cardinals (see Definition 1.22) are much stronger than measurable cardinals, being in principle inaccessible limits of $H(\lambda)$-strong cardinals introduced above (for certain λ's). However, a Woodin cardinal may not even be weakly compact (while it is a Mahlo cardinal). Its classification is still (R$^-$) and (L), as will be apparent from Theorem 2.18.

The following theorem appears in [3] as Theorem 4.5:

Theorem 2.17. *Let V satisfy* GCH *and let F be an Easton function respecting Ramsey cardinals. Let A_0 be the class of all Ramsey cardinals. Then in any generic extension $V[G]$ by P_F, F is realised and the set of Ramsey cardinals contains A_0.*

We should note that the proof of Theorem 2.17 utilizes a characterisation of Ramseyness by means of elementary embeddings, to apply an appropriately tailored lifting argument.

The following theorem appears in [2] as Theorem 1:

Theorem 2.18. *Let V satisfy* GCH *and let F be an Easton function respecting Woodin cardinals. Let A_0 be the class of all Woodin cardinals. Then in any generic extension $V[G]$ by a certain cofinality preserving forcing P, F is realised and the set of Woodin cardinals contains A_0.*

The forcing P in the statement of the theorem contains the α-Sacks forcing at the critical stages (regular closure points α of F), similarly as we discussed below Theorem 2.15. The key lemma for the preservation of Woodiness is Lemma 14 in [2].

We now turn to supercompact cardinals. The first generalisation of the Easton theorem for large cardinals actually appeared for the supercompact cardinals, see [17]. Since supercompact cardinals have reflection properties, it is not possible to realise every F and preserve supercompact cardinals; Menas identified a property of F which is sufficient for preservation of supercompact cardinals:

Definition 2.19. *An Easton function F is said to be locally definable if the following condition holds:*

There is a sentence ψ and a formula $\varphi(x, y)$ with two free variables such that ψ is true in V and for all cardinals γ, if $H(\gamma) \models \psi$, then $F[\gamma] \subseteq \gamma$ and

$$\forall \alpha, \beta \in \gamma (F(\alpha) = \beta \Leftrightarrow H(\gamma) \models \varphi(\alpha, \beta)). \tag{2.6}$$

The following is a theorem in section 18 of [17]:

Theorem 2.20. *Let V satisfy* GCH *and let F be a locally definable Easton function respecting supercompact cardinals. Let A_0 be the class of all supercompact cardinals. Then in any generic extension $V[G]$ by the forcing P_F of Definition 2.4, F is realised and the set of supercompact cardinals contains A_0.*

The theorem is proved using a "master condition" argument[15] applied to the forcing, which makes it possible to use Cohen forcing at closure points of F; compare with the discussion below Theorem 2.15. Theorem 2.20 was generalised also for the strong cardinals (see Definition 2.13); see [10, Theorem 3.17].

Theorem 2.21. *Let V satisfy* GCH *and let F be a locally definable Easton function respecting strong cardinals. Let A_0 be the class of all strong cardinals. Then in any generic extension $V[G]$ by a certain cofinality-preserving forcing P, F is realised and the set of strong cardinals contains A_0.*

The forcing P contains the α-Sacks forcing at regular closure points α of F.

Let us conclude this section by remarking that there are results similar to these theorems which are formulated for a λ-supercompact cardinal κ which is also $H(\nu)$-strong for some $\lambda < \nu$; see [11, 4].

2.5 Open questions

Considering the variety of large cardinal concepts, it is no surprise that many of them have not been studied from the point of their compatibility with patterns of the continuum function. For instance the following cardinals have not been studied:[16]

- While strong compactness is close to supercompactness in the consistency strength, the dropping of normality of the witnessing ultrafilter makes it less well-behaved. In particular, a characterisation by means of an elementary embedding only gives the following (compare with Definition 1.21):

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[15] Roughly, in order to lift an embedding j between transitive classes M and N, the pointwise image of a P-generic filter g, $j''g$, is an element of N, generating a suitable $j(P)$-generic filter h over N containing $j''g$. $j''g$ is called a master condition. In crucial situations, $j''g$ is usually too big to be in N; a typical case where $j''g$ is in N is when j is a supercompact embedding. There is no master condition for arguments starting with $H(F(\kappa))$-strong cardinals.

[16] To our knowledge, no complete generalisation of the Easton theorem has been formulated yet.

Definition 2.22. *Let κ be an inaccessible cardinal and $\lambda > \kappa$ a cardinal. κ is λ-strongly compact if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and for any $X \subseteq M$ with $|X| \leq \lambda$ there is $Y \in M$, $Y \supseteq X$, such that $M \vDash |Y| < j(\kappa)$.*

These weaker properties of the embedding suggest a different lifting method – instead of lifting an embedding, one can lift directly the ultrafilter (as in [13], albeit in a different context).

- We say that κ is a *Shelah cardinal* if for every $f : \kappa \to \kappa$ there is $j : V \to M$ with critical point κ such that $V_{j(f)(\kappa)} \subseteq M$. Very little has been published about this cardinal with respect to the continuum function.
- Rank-to-rank embeddings (the hypotheses I3–I0). A partial result appeared in [8].

There are many other cardinals which can be studied, so our list is far from complete.

3. In the converse direction

In the whole paper, we studied the question of preserving large cardinals while manipulating the continuum function. As a curiosity, we show in this section that by manipulating the continuum function, it is possible to wipe out all large cardinals.

Theorem 3.1. *Let $M = V_\kappa$, where κ is an inaccessible cardinal. Suppose $I = \{\alpha < \kappa \mid \alpha \text{ is inaccessible}\}$ is a non-stationary subset of κ. Then there is a forcing P of size κ, definable in $H(\kappa^+)$ such that in $M[G]$, there are no inaccessible cardinals, for any P-generic filter G over V.*

Proof. Let C be a club disjoint from I, and let $\langle c_i \mid i < \kappa \rangle$ be the increasing enumeration of C. Define P to be a product of Cohen forcings with Easton support as follows: define $Q_i = \mathrm{Add}(c_i^+, c_{i+1})$ for $0 \leq i < \kappa$, and $Q_{-1} = \mathrm{Add}(\aleph_0, c_0)$. Set

$$P = \prod_{-1<i<\kappa}^{\text{Easton}} Q_i,$$

where the superscript "Easton" denotes the Easton support.

Let G be a P-generic filter over V. By definition of P, if $\mu < \kappa$ is a limit cardinal closed under the continuum function in $V[G]$, then $\mu \in C$. Since $C \cap I = \emptyset$, it implies that in $V[G]$ there are no inaccessible cardinals below κ.

Finally, since κ is still inaccessible in $V[G]$, $M[G]$ is a transitive model of set theory without inaccessible cardinals as desired. □

Note that if M satisfies GCH, then the forcing P preserves cofinalities.

To destroy all inaccessible cardinals in M, it suffices to find a forcing which forces a club disjoint from inaccessible cardinals. The idea comes from [7].

Theorem 3.2. *Let M be as above. There is a forcing P of size κ which does not change $V_\kappa = M$ such that in $V[G]$ there is a club $C \subseteq \kappa$ disjoint from I, the set of inaccessibles below κ.*

Proof. Let conditions be functions from ordinals $\alpha < \kappa$ to 2 such that if $\beta < \kappa$ is inaccessible, then $\{\gamma \in \mathrm{dom}(p) \cap \beta \mid p(\gamma) = 1\}$ is bounded in β.

The forcing is κ-distributive because it is κ-strategically closed. So V_κ is not changed, and consequently all cardinals $\leq \kappa$ are preserved. Moreover since κ is inaccessible, P has size κ, so all cardinals are preserved.

Clearly, if G is P-generic over V, then

$$A = \lim\{\alpha < \kappa \mid (\exists p \in G)(p(\alpha) = 1)\}$$

is a club disjoint from I. $\square$

Remark 3.3. Note that the same proof can be rephrased as turning a Mahlo cardinal into a non-Mahlo inaccessible cardinal.

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