Contribution of Geometry to the Goals of Education in Mathematics[1]

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Abstract: Through visualisation, geometry can mediate understanding of some demanding arithmetic and algebraic concepts, relationships, processes and situations for pupils. This thesis is explained by the method of genetic parallel and of a didactic analysis of two educationally interesting problem situations. Theoretical considerations are illustrated by several real experiences. Suggestions for the application of theoretical results are given in conclusion.

Keywords: method of genetic parallel, language of arithmetic and algebra, *shaped psephophory*, the language of geometry, visualisation, gnomon, discovery, proof, process, concept

1 Introduction and Methodology

Geometry appears at two levels in school mathematics. At the first level, plane and space shapes, relationships, constructions, proofs, etc. are introduced to pupils. At the second level, geometry provides support for arithmetic and algebra. Pupils can be strongly dependent on the visualisation of arithmetic and algebra. For example, some pupils are able to understand the additive structure of integers already in the second grade of the primary school with the help of a number line, but without it, they cannot carry out additive operations with negative numbers even in the eighth grade. The goal of the study is to provide examples of how geometry can help in understanding arithmetic or algebraic concepts, processes, relationships and arguments. The geometric support is decisive for some pupils, not only from the point of view of understanding the subject matter but also from the point of view of their approach to learning. A superficial approach which enables pupils to "meet the requirements of knowledge reproduction" changes due to visualisation into a deep approach which enables them to "really understand the subject matter" (Mareš, 1998, p. 39).

Two methods are used in this study. The first is the method of genetic parallel which postulates that relations found in the phylogeny are inspirations for revealing relations in the ontogeny. An inspiration for the use of the second level of geometry in the teaching of mathematics can be found in the sixth century BC when the

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language of shaped psephophory shifted mathematical thinking from the area of arithmetic to geometry. We also use the results gained by the study of mathematical languages in history by L. Kvasz (2008).

The second research method consists of the didactic analysis of problem situations. For each such situation, arithmetic, algebraic and geometric approaches are shown and didactic phenomena analysed. Short stories which illustrate the analyses come from our archives.

Finally, some suggestions for the use of theoretical results in practice are given.

2 Method of Genetic Parallel

The following quotation aptly characterises the idea of genetic parallel:

"The growth of the tree of mathematical knowledge in the mind of one person (ontogeny of mathematics) will only be successful if it replicates to a certain extent the history of the development of mathematics" (Erdnijev, 1978, p. 197).[2] This idea is expressed more precisely by Freudenthal (1991, p. 48): "Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now." (Freudenthal, 1991, p. 48). Jankvist (2009) illuminates the method of genetic parallel in a concise way.

Mathematical knowledge of pre-Greek civilisations from the Nile and Hindu basins and Mesopotamia answered the question *How?*. How can we calculate? How can we find? How can we construct? … It was mainly the knowledge of calculation, in the present language, arithmetic knowledge. The Greeks were the first to ask for the basis (úsia) and cause (*aiton*, *aitia*) of things and phenomena. They understood that the knowledge of causes is more important than the knowledge of instructions:

> […] we suppose artists to be wiser than men of experience (which implies that Wisdom depends in all cases rather on knowledge); and this because the former know the cause, but the latter do not. For men of experience know that the thing is so, but do not know why, while the others know the 'why' and the cause. (Aristotle, The Metaphysics I, p. 1)

Milesian philosophers in the sixth century BC explored the basis, the substance of the world. They found it in water, air or indefinite *apeiron*. Pythagoras claimed number to be the essence of the world. "Pythagoras, …, said that 'all things are numbers'. This statement, interpreted in a modern way, is logically nonsense, but what he meant was not exactly nonsense." (Russell, 1965, p. 54). Pythagoras believed that all phenomena in the world such as joy, truth, justice, courage, male principle, female principle, etc. had their own representations in the world of numbers and thus the relations of the world were depicted in the relationships among numbers. At present, we would say that the unclear and variable scheme of things is grasped

2 Рост древа математических знаний в голове отделного человека (онтогенез математики) будет успешным тогда, когда он повторяет в известной мере историю становления этой науки (филогенез математики).

by a strict and immaculate scheme of numbers. The knowledge of eternal essences, scientific knowledge (*epistémé*), is more important than practical knowledge of counting (*phronesis*).[3]

In the Pythagorean school, mathematics as a scientific discipline was born, as a discipline looking for exact definitions of concepts, general regularities and their proofs. An important by-product, possibly the key one, of this birth was the change of language. The language of pebbles (*pséfoi*), which was used for counting in the whole Mediterranean in the sixth century BC, was changed into the language of shapes (*shaped psephophory*; in the contemporary terminology figurative numbers). The number previously represented by a pile of pebbles was represented by a shape (made by spreading the pebbles into this shape). In this way, figurate numbers originated (Figure 1).

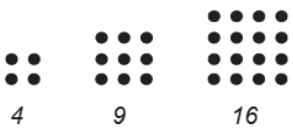

Figure 1a Square numbers

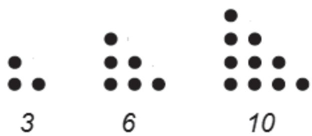

Figure 1b Triangular numbers

In this language, the square represents an infinite sequence of numbers; the same applies for the shape of a triangle. On the other hand, all even numbers can be described by a single shape, a rectangle of the width 2 and of any length; in brief *2-rectangle* (Figure 2a). Each odd number can be described by a *2-rectangle with an appendix* (Figure 2b).

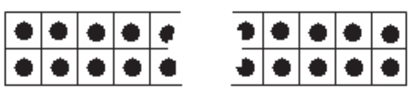

Figure 2a An even number of pebbles arranged as a 2-rectangle

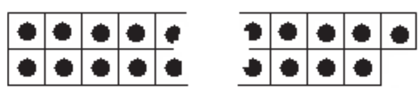

Figure 2b An odd number of pebbles arranged as a 2-rectangle with an appendix

The new language made it possible to formulate and prove general statements which went beyond the horizon of the then illuminated world of numbers. For example, the statement

The sum of two odd numbers is an even number. (*)

does not only hold for small numbers which we can imagine but also for numbers inconceivably big, such as the number of grains of sand in the desert. If we connect two odd numbers (i.e., two 2-rectangles with appendices), the two "appended" pebbles make a pair and the result is a 2-rectangle, that is an even number. The connection is in Figure 3.

3 F. Korthagen (2011, p. 36–44) elaborates this typology in more detail.

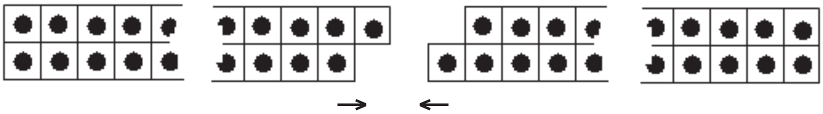

Figure 3 The connection of two odd numbers results is a 2-rectangle

The presented proof of the statement (*) holds for all numbers. That is guaranteed by the fact that the length of the 2-rectangle with an appendix (both right and left ones) can be of any measure.

It is interesting to note that Pythagoras could have stated the same idea without the use of geometry. He could have defined an even number as any number of pairs of pebbles and an odd number as any number of pairs of pebbles plus one pebble. Why did Pythagoras not do so? Why did he introduce the language of geometry? Most probably it was because the arithmetic pairing was understood in a procedural way as manipulation, which appears to be unrealisable for big numbers. The geometric shape is understood in a conceptual way as a ready-made object which exists regardless whether we can or cannot imagine it; an object which we know through its properties, not through sensory experience; an abstract object. And thus the birth of mathematics is connected to the change of language: the number, previously understood in a procedural way as a pile of pebbles, is now understood as a concept, as a geometric shape.

The change of language is a key constituent of the historical development of mathematics. L. Kvasz (2008, p. 16) presents six points of view from which he studies the development of the languages of mathematics:

1. logical power – how complex formulas can be proven in the language;
2. expressive power – what new things can the language express, which were inexpressible in the previous stages;
3. explanatory power – how the language can explain the failures which occurred in the previous stages;
4. integrative power – what sort of unity and order the language enables us to conceive there, where we perceived just unrelated particular cases in the previous stages;
5. logical boundaries – marked by occurrence of unexpected paradoxical expressions;
6. expressive boundaries – marked by failures of the language to describe some complex situation.

If we apply the first five points on the birth of shaped psephophory as a new language, we can see that there is a series of cognitive and meta-cognitive shifts:

1. from items to generality and from the concrete to the abstract;
2. from the work with numbers before the horizon (small numbers) to the work with any number;
3. from the practical relationships to the theoretical ones (from *phronesis* to *epistémé*);

4. from a set of instructions of calculation, to a systematically shaped psephophory (in other words, from a process to a concept);
5. from sensory evidence to general proofs.

3 Didactic Analyses of Problem Situations

Let us now consider the second of the research methods used, the analysis of tasks which allow for both arithmetic and geometric solutions. We will show how predominantly procedural arithmetic or algebraic solutions become more understandable by the use of geometric concepts. Two illustrations will be given.

3.1 Task 1: *Prove statement (*)*.

In the arithmetic language, the parity of the number can be found through the record of the number in a decimal system: number n is even *iff*[4] its last digit is 0, 2, 4, 6, or 8; number n is odd *iff* its last digit is 1, 3, 5, 7, or 9. Let us add, that the positional decimal system was discovered more than 1500 years after Pythagoras. Nevertheless, our pupils know this powerful language and already in the third grade most of them also know that the parity of the number is given by its last digit.

An arithmetic proof of the statement (*) lies in checking the following fact: the sum of any two numbers from 1, 3, 5, 7, 9 equals the number which ends with the digit 0, 2, 4, 6, or 8. This process cannot be seen as one whole, it is necessary to keep a step by step record of it. The proof of the statement (*) requires checking 15 simple sums.

It is a mathematically correct proof which can be independently discovered by pupils from grade 4.

In the algebraic language, the parity of the number is most often given by the following characterisation: the number is even *iff* it can be written in the form of $2k$, where k is a natural number; the number is odd *iff* it can be written as $2k + 1$, where k is a natural number.

The algebraic proof of the statement (*) lies in the manipulation with the expression $(2k + 1) + (2m + 1)$ into the form of $2 \times (k + m + 1)$, that is, into the identity

$$(2k + 1) + (2m + 1) = 2 \times (k + m + 1), \qquad (**)$$

where k, m are natural numbers and thus $k + m$ is also a natural number.

The identity (**) is a concept and the proof is a correct one. We know that the proof is not understandable for many pupils of grade 9 but we also know of cases when the proof was discovered by a pupil from grade 6.

In the geometric language of shaped psephophory, the parity of a number is given by fig. 2a and 2b and the proof of the statement (*) by fig. 3. The proof is correct and pupils from grade 4 are able to discover and understand it.

[4] I.e. if and only if.

The didactic analysis of the above three proofs of the statement (*) looks into the difficulty of grasping the concepts of even and odd and into the thinking processes present in the proofs.

The arithmetic grasping of the concept of even and odd is based on a two-step process: (1) a pupil realises that for a number to be even or odd, the last digit is the key one, (2) he/she finds out if this number belongs to the set {0, 2, 4, 6, 8}, or to the set {1, 3, 5, 7, 9}. This characterisation of evenness/oddness does not have to be evident for a pupil from grade 2.

Story 1

Five pupils from grade 2 were to find an odd number whose last digit was 4. One boy started to laugh and the others immediately reacted that it was not possible. After a while, Adela said that it would have to be very big.

The story shows that the knowledge “the parity of the number is given by its last digit” is not evident for all pupils in grade 2.

The arithmetic proof is a lengthy one. It is redundant for some pupils because the situation is clear to them. However, some pupils feel the need to verify all 15 sums.

Story 2

Adela from story 1 created Table 1. She was looking at it with delight for some time and then she said “yes, now it is clear”. The next day, she brought two more tables to the teacher. The first one was for the proof of the statement that the sum of two even numbers is an even number and the second for the proof of the statement that the sum of an odd and even number is an odd number. What led Adela to create the table? She felt that the series of individual calculations did not bring an insight into the proof and found the right way to acquire it. A long process of the creation of the table led to the table as a concept in her mind. Thus the table became the main bearer of the proof of the statement (*) for her. This proof, as well as the following algebraic one are axiomatic proofs from the point of view of Housman and Porter’s classification (2003). The marked difference between them is described within Kvasz’ theory (2008) here by the language used.

Table 1 Addition of odd numbers

+	**1**	**3**	**5**	**7**	**9**
1	2	4	6	8	10
3	4	6	8	10	12
5	6	8	10	12	14
7	8	10	12	14	16
9	10	12	14	16	18

The algebraic grasping of the concept of even and odd is based on understanding the notation of $2k$ and $2k + 1$, where k is a natural number. This characterisation of parity is not understandable for many pupils; often, they simply remember it as is illustrated by the following story.

Story 3

In grade 7, Bara discovered the algebraic proof (**). When asked by the teacher, she showed the discovery to the class. In the subsequent discussion in the class, the following statements could be heard (among others):

Cyril: What is odd then? It is $2k + 1$, or $2m + 1$?
Dana: But I found out that even (she points to the record $2 \cdot 3.5 - 1 = 7 - 1 = 6$).
Ema: But last time, an odd number was $2k - 1$.
Philip: And can I write it as $1 + 2k$, too?

The algebraic proof is brief and for a pupil who understands the identity (**) as a concept, it is clear. If the pupil does not understand the meaning of records $2k$ and $2k + 1$, he/she cannot understand the proof either. These are all four pupils from story 3 and many others from grade 7. However, even the pupils who have a good understanding of odd and even numbers in the language of letters are often not able to grasp the proof as a whole. It is illustrated by a story of one of our colleagues AŠ.

Story 4

AŠ speaks about her experience from grade 7: *I know that it is difficult, therefore I took great care that all knew how to record an even number and how to record an odd number. Then I let Gita, who is the best mathematician from the class, to make the sum* (**). *The girl did it marvellously. I asked the class if they understood. All nodded that they did. So I asked them to come and show in a similar way that the sum of three odd numbers is odd. Only two pupils put their hands up and shyly one more. I know about them that they can do it but beside them, none. It is simply too difficult for seven graders.*

Why is the algebraic proof so difficult? There are two causes. The first and key one lies in the fact that the teaching of algebra concentrates mostly on the manipulation with algebraic expressions, that is, the manipulation with the letters. Little attention is paid to the clarification of the meaning of this language.

Story 5

In grade 8, Hanka is solving a task at the blackboard:

How long does it take a cyclist who goes at the speed of $v = 16$ km/h to cover the distance $s = 10$ km?

After a while, the girl says: "I have forgotten the formula." The teacher writes $v = s/t$ on the board. The girl says: "I know it and I also know that $s = v \times t$, but I have forgotten the third one." The teacher is lost for a moment what to do and then she asks the class: "Could anyone help Hanka but without actually saying the formula?" Ivan says: "Hanka, write the equation $4 = 3/x$ and solve it." Without any problem, Hanka finds $4x = 3$ and $x = ¾$. Ivan continues: "Now you write $a = b/x$ and solve it." Again Hanka finds $x = b/a$. Ivan: "Excellent. Now, write $v = s/x$ and when you solve it, use letter t instead of x." Hanka writes the equation, compares it to what she has been doing earlier, laughs and without solving anything she says "Yes, I have remembered," and writes $t = s/v$.

In the context of equations, the girl had no problems with the algebraic manipulation but she had no clue that the same manipulation could also be made in a different context. The story shows an inappropriate approach to the introduction of letters to pupils. The manipulation with letters has no support in semantic ideas. In the case of proof (**), it is possible to find such a support just in the psephophory geometric proof as Figure 4 depicts.

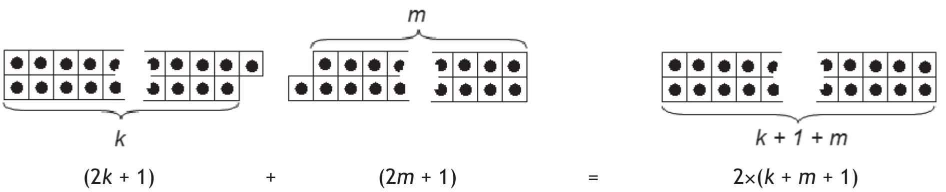

Figure 4 Geometric proof

The above didactic solution of the problem how to mediate the understanding for the identity (**) for pupils is similar to the didactic solution of the problem how to teach pupils to solve word problems. For word problems, dramatization helps, that is, mathematical phenomena are grasped in a semantic way, they are supported by a pupil's real life experience. We have used visualisation for the identity (**), that is, we supported algebraic objects by a pupil's geometric experience. In both cases, we can speak about the semantic way of grasping the situation if we suppose that geometric objects we work with are for pupils evident in the same way as life experience at word problems. The pupil who does not have this evident knowledge of geometric shapes such as rectangle and shapes on a square grid cannot use the way via visualisation of the relation (**) successfully. He/she is in the same situation as a pupil who is solving a word problem about dairy cows but does not know the word dairy cow.

We have finished our considerations about the first task. The second task often appears in secondary textbooks.

3.2 Task 2: Find the sum of the first n odd numbers: $s_n = 1 + 3 + 5 + … + 2n - 1$.

The arithmetic solution is based on the experiment, observation and generalisation. The pupil finds out that $s_1 = 1$, $s_2 = 4$, $s_3 = 9$, $s_4 = 16$, … and notices that the given numbers are squares and that in all the cases, the result is $s_n = n^2$. Thus

$$s_n = 1 + 3 + 5 + … + 2n - 1 = n^2. \qquad (***)$$

The pupil considers this generalisation of four observed cases to be the solution. He/she might check it by two or three more calculations. Even though this solution is not proper, the result is correct and the pupils believe it. This proof is a typical example of inductive reasoning which is characterised by Housman and Porter (2003) as: "A student with an inductive proof scheme considers one or more examples to be convincing evidence of the truth of the general case." (p. 40)

The algebraic solution is based on the manipulation of expressions. If the pupil adds the first and last elements, i.e., $1 + (2n - 1)$, the second and the last but one elements, i.e., $3 + (2n - 3)$, etc., he/she finds out that each from the sums is $2n$ and that if n is an even number, there are $n/2$ of these sums. Thus for n even, it is $s_n = 2n \times n/2 = n^2$. Then the pupil finds out that for n odd, the result is the same.

In both cases, arithmetic and algebraic ones, the pupil gets to the result of $s_n = n^2$. However, the mathematics teacher is sometimes not satisfied with this result and asks for the proof. Few pupils understand what the teacher is looking for. Such a pupil shows the proof by mathematical induction[5], which the teacher is satisfied with but most pupils, in fact, do not know what it is about.

The geometric solution is based on the visualisation of the expression. An important role is played by the shape which the Greeks called *gnomon* and which was commonly used by Euclid. We will explain it. If we cut out a square from another square so that both squares have a common vertex, then the remaining shape was called gnomon by the Greeks. If the side of a bigger square is by k (pebbles) bigger than the side of the smaller square, we say that this gnomon is of width k, in brief k-gnomon. Figure 5a shows 1-gnomon and Figure 5b 2-gnomon. We can see that each 1-gnomon is an odd number and each odd number is 1-gnomon. Thus, the sum of several odd numbers is made of several gnomons.

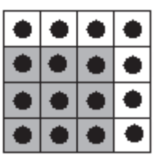

Figure 5a 1-gnomon

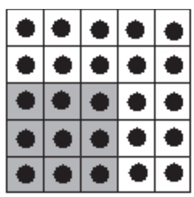

Figure 5b 2-gnomon

Figure 6 shows the solution to the task by shaped psephophory. The figure is made like this: first, one pebble is placed, with 3 pebbles around it in the shape of 1-gnomon, there are 5 pebbles in the shape of 1-gnomon around this square of 4 pebbles, etc. It is clear that by gradual adding of 1-gnomon, the resulting square "grows" but remains the square.

[5] Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true for all natural numbers (positive integers). If $V(n)$ is a statement which is true for all natural n, then we can prove that $V(n)$ holds for all natural numbers by proving the statement $V(1)$ and the implication $V(k) \Rightarrow V(k+1)$ for all natural k.

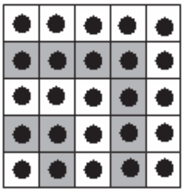

Figure 6 Solution by adding of 1-gnomon

Even against this geometric solution we can protest by saying that the statement has not been proved. The proof would be made by mathematical induction again, this time in the context of shaped psephophory.

The proof will be described in a hypothetical dialogue between two pupils.

Story 6

Jana, which discovered the relationship $s_n = n^2$ in class in an arithmetic way, boasted about her success to her brother. The brother showed her how to derive the relationship from fig. 6. Jana liked it and asked the teacher to be allowed to show it in class. The class liked the figure, too, but Karel protested: “But what if the next gnomon does not fit the preceding square for big numbers? What if it is a bit shorter or longer?” The teacher suggested that the pupils thought about this objection. Jana: “Let us presume that until number k it will work. So the sum of the first k odd numbers is a square of side k. What 1-gnomon follows? The last gnomon had $(2k - 1)$ pebbles and that is why the following has $(2k + 1)$ pebbles. And that is just what we need to get a square of side $k + 1$ from the square of side k.”

Jana’s proof is based on mathematical induction. As it is an illustrative one, pupils will understand it better than the algebraic one.

4 Conclusions

The study analyses in a didactic way two tasks in which geometry markedly helps understanding an arithmetic situation. A deeper consideration of the cognitive structure of this didactic problem shows that geometry helps to transform arithmetic or algebraic thinking of a procedural nature to the conceptual level. In task 1, the proof was divided into many partial steps in the arithmetic language and too sophisticated in the algebraic language to be grasped by most of the pupils. In the geometric language, the proof was short and clear. Task 2 concentrated on finding the formula $s_n = n^2$. The arithmetic language leads a perceptive pupil to the solution via isolated and generic models (Hejný, 2012). However, it does not provide persuasive argumentation. Again, the algebraic language requires sophisticated considerations. The geometric language shows in a single picture both the process of the growth of the resulting square and the argumentation described in story 6. In this story, we can also see the geometric contribution on a meta-cognitive level: the proof by mathematical induction, which is in its arithmetic or algebraic realisations non-understandable to most of the pupils, is much easier to grasp in a geometric context.

We should add that the visual support of arithmetic and algebra can be widened by dramatization for kinaesthetic support. For example, task 3 can be solved by pupils from grade 4 by walking on a staircase with numbered steps (i.e., on a number line).

Similar didactic analyses as the above for tasks 1 and 2 can be made for many different tasks such as:

Task 3. Solve the equation $|1 - |x + 1|| = 2$.

Task 4. Find the sum $s_n = 1 + q + q^2 + \dots + q^n$, where n is a natural number and $q > 0$ is a real number.

Task 5. Find the sum of an infinite series $s_n = 1 + q + q^2 + \dots + q^n$, where $0 < q < 1$.

Task 6. Prove that $\sin\alpha + \sin\beta = \sin\alpha \cos\beta + \cos\alpha \sin\beta$, for $0 < \alpha$, β and $\alpha + \beta < \pi/2$.

Our experience as well as the experience of many cooperating teachers show that visualisation is a crucial scaffold for securing understanding of concepts, relationships, situations and processes in mathematics for some pupils. The problem of the use of geometry for the understanding of phenomena in arithmetic, algebra, but also combinatorics, probability or logic is elaborated in many publications. From one of them, which has become a classic, we choose one fitting verse to conclude with:

Geometry is to open up my mind
so I may see what has always been behind.
Henderson (2001, p. *ii*)

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