Exploring the Cognitive Dimension of Teaching Mathematics through Scheme-oriented Approach to Education[1]

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Abstract: Following H. Freudenthal, E. Fischbein and many other mathematics educators, we consider the main way to improve the teaching of mathematics lies in changing teacher's educational styles to develop pupils' creativity and intellectual autonomy. In our interpretation, this means using scheme-oriented education based on the theory of generic models we describe. On the basis of these ideas, a tool for the analysis of teachers' teaching style is developed. The tool and its application are described in Jirotková's article in this issue.

Keywords: qualities of teaching mathematics, teachers' pedagogical beliefs, scheme-oriented education, theory of generic models, reflection, teaching style, transmission and constructivist educational styles, goals of mathematics teaching

1 Rationale and Formulation of the Problem

This study is, in a sense, a contribution to the on-going discussion in the society about the quality of the teaching of mathematics at schools, stemming from a several year long decline of Czech pupils' results in international studies of TIMSS and PISA. Even though much criticism of drawing hasty conclusions from these studies has appeared (Štech, 2011), results of McKinsey's study (2010) clearly call for the improvement of teaching mathematics in the Czech Republic and claim that "… it is necessary to change attitudes and behaviour of people – in this case of more than 100,000 teachers – and that is a task extremely complicated for any institution." (McKinsey & Co., 2010, p. 4) After more than forty years of investigating the teaching of mathematics, we see the main deficiency to be the focus on reproductive and imitative activities in connection with a low representation of pupils' creative activities in teaching. This deficiency has been around for a long time and many documents of the Ministry of Education as well as research studies have been pointing out the necessity to enhance creativity in the teaching of mathematics for more than fifty years.

From the late fifties until the early eighties of the last century, there was an attempt in many developed countries to bring creativity to mathematics lessons by curriculum change (Adler, 1972; Hilton, 1977). This initiative under the name New Math came from the USA where "Sputnik crisis" caused a pressing need to enhance

42 the professional level of engineers in the country. At the same time, A. N. Kolmogorov (1973) in the USSR and such authorities as G. Papy (1963), J. Dieudonné (1961), H. Freudenthal (1973), R. Thom (1973) in Western Europe urgently required changes in the teaching of mathematics at primary and secondary schools. The idea of changing the curriculum was the strongest. Its advocates believed that if a traditional curriculum dealing with algorithms of calculations was replaced by an approach based on set theory, the transmissive way of teaching[2] would change into a dialogic and creative one. The expected change indeed came (Kabele, 1965/1966; Kabele, Kníže, 1969; Hruša, 1968/1969). The new curriculum was very successful mainly in pupils' motivation. Previous fear of mathematics was replaced by enthusiasm of both pupils and teachers. Mathematics became one of the most popular subjects. However, the enthusiasm lasted only a few years. Gradually, the creative zeal of teachers and pupils diminished and drill came back to schools. In 1973 in the USA, M. Kline's article *Why Johnny Can't Add: The Failure of the New Mathematics* appeared and presaged the end of the set initiative.

In the former Czechoslovakia, sets were introduced to schools in 1976 and wound down in the nineties. A new set of textbooks only appeared after 1990, however, this did not lead to a marked improvement of the teaching of mathematics, rather to the contrary, as D. Greger (2011) comments on a general level.

Why did the journey of set mathematics in all the countries where it was introduced have such a dramatic course? The answer will allow us to learn an important lesson from this extensive experience. First of all, the introduction of sets was frustrating for many teachers as they had to learn new things. They could no longer instruct pupils as there were no instructions to give. They had to lead discussions, solve various problems, being creative. They became the bearers of the climate of creativity. They themselves looked for answers and they conveyed their curiosity to their pupils. And in return, pupils' unprecedented activity motivated teachers. All looked bright. But when the teachers after a couple of years mastered the subject matter, they started to economize their work and create conventions and rules as a supposed tool for the more effective educational process. Curiosity and joy from discovery left mathematics lessons. Pupils were again required to reproduce and imitate. Fear and boredom returned to mathematics – fear of those who could not work at such a required pace and boredom of those who felt the need to solve markedly more difficult problems.

Set based curricula have left schools in all the countries where it was introduced, including the Czech Republic. It was known that the return to the previous situation would not solve the original problem how to stress creativity and lessen memorisation in the teaching of mathematics. New initiatives have proceeded in several directions. Our research belongs to those following Hans Freudenthal's motto: "Mathematics is a human activity." We try to bring to life ideas which he formulated more than forty years ago: "Mathematics … is an activity of solving problems, of looking

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[2] The term transmissive education is used in the sense of Askew at al. (1997).

for problems, but it is also an activity of organizing a subject matter. This can be a matter from reality which has to be organized according to mathematical patterns if problems from reality have to be solved. It can also be a mathematical matter, new or old results, of your own or others, which have to be organized according to new ideas, to be better understood, in a broader context …" (Freudenthal, 1971, p. 413–414).

Worldwide experience with New Math has shown that understanding mathematics is not given by the content but rather by the method of teaching. This had led to deep investigation of pupils' cognitive processes, see, for example, Sfard (1991), Schonfeld (1992), Dubinski, McDonald (2001), Hershkowitz, Schwarz & Dreyfus (2001). Many important research results have not found their way into practice. Thus, research in mathematics education has focused on the teacher's personality, the teacher being the main actor in the teaching process. The teacher's value system and his/her pedagogical beliefs have come to the fore, as the main parameters of the quality of teaching.

In the last twenty years, teachers' beliefs have been investigated in many studies in mathematics education. Their philosophical underpinning can be found in the work of P. Ernest (1991) and, for example, A. Thompson (1992). We focus only on those studies which explore the relationship between teachers' beliefs and teachers' practice and who are most often quoted in the literature i.e. Törner, Pehkonen, Goldin (i.e., proceedings edited by Leder et al., 2002), Speer (2008) and many others.

A teacher who feels that a traditional transmission style (see below) is ineffective and looks for a more effective one will improve his/her work if such a style is offered to him/her. However, according to our experience, there is a lack of such teachers. The teacher who is satisfied with his/her transmissive teaching must be persuaded that his/her work will be more attractive for him/her, more successful for his/her pupils and more joyful for everybody when drill is suppressed and creativity enhanced which develops a pupil's mathematical thinking. The consequence of this observation is a challenge: Look for ways to shift the teacher's beliefs so that he/she stresses creative aspects and tones down transmission aspects in his/her teaching.

A change of a person's beliefs intervenes with his/her hierarchy of values and any shift is difficult, sometimes even impossible. Our experience shows that not all teachers can be influenced and that ways to shift pedagogical beliefs can vary from teacher to teacher.

The goal of the study is to look for answers to the following three questions which concern the teaching of mathematics and an effort to shift the teacher's educational style towards a constructivist one[3]:

1. Which phenomena determine the quality of the teacher's work in teaching mathematics?
2. How to identify the teachers for whom a shift of the educational style seems to be promising (hopeful)?

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[3] The term constructivist education is used in the sense of (Noddings, 1990).

3. How to find out which ways can lead to the required change for a particular teacher?

2 Cognitive Goals of the Teaching of Mathematics

In view with Freudenthal's (1973), Fischbein's (1999) and many others' ideas, we characterise effective teaching of mathematics by three cognitive goals:
– a pupil understands mathematics, his/her knowledge is not mechanical;
– a pupil is intrinsically motivated for work, he/she is not frustrated by mathematics;
– a pupil develops intellectually; by that, we mainly mean the development of the ability to: 1. communicate mathematically both orally and in writing, 2. cooperate in a group or even lead a group to solve mathematical problems, 3. analyse a mathematical problem situation, 4. effectively solve mathematical problems and 5. correct one's own mistake.

2.1 Understanding mathematics

A pupil who knows a formula, a rule or a definition or a pupil who can calculate quickly and precisely but cannot answer the question *why* his/her procedures work has mechanical knowledge only. Typical features of such a prosthetic piece of knowledge are: discontinuity from other pieces of knowledge, quick forgetting, inability to self-correct mistakes. A pupil with superficial knowledge of mathematics can be successful in standard tests but cannot solve non-standard problems and does not understand mathematics.

Thus for teachers it is important to know how a pupil comes to understand mathematics and how it is possible to diagnose this understanding. Both stem from the mechanism of a concept development process. The one which will be used here was outlined more than fifty years ago by Vít Hejný (1942). His ideas were further developed in the Bratislava seminar of didactics of mathematics and the new results were continuously tested in several classes from 1975 to 1989. Since 1992 this theory is developed by the work of the author and his collaborators at the Faculty of Education, Charles University in Prague.

The building blocks of learning with understanding are *generic models* which, in a person's mind, create a complex multi-layered dynamic structure organised into *schemes*. Both terms will be explained below.

2.2 Theory of generic models

Our model of the process of gaining knowledge is based on five stages. It starts with motivation and has as its core two mental shifts: the first leads from concrete knowledge (isolated models) to generalised knowledge (generic knowledge) and the

second from generic to abstract knowledge. The permanent part of this process of gaining knowledge is that of crystallisation, which involves integrating new knowledge into the already existing mathematical structure (Hejný, 2011a).
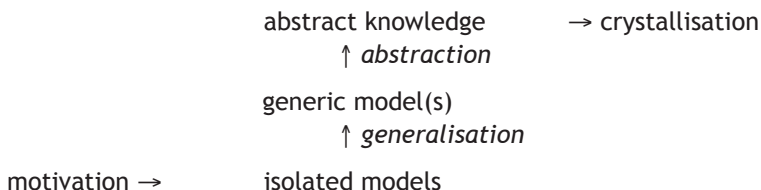
The whole process is shown in figure 1.

abstract knowledge   → crystallisation
        ↑ *abstraction*

generic model(s)
        ↑ *generalisation*

motivation →   isolated models

Figure 1 Scheme of the process of gaining knowledge according to the Theory of Generic Models

Only the first three stages will be explained in detail because they are essential at elementary school level.

Motivation. We see motivation as the tension which occurs in a person's mind as a result of:

– the need to repeatedly experience joy from intellectual success which comes after solving a problem or even the discovery of a new truth, and
– the discrepancy between the existing and desired states of knowledge. The discrepancy comes from the difference between 'I do not know' and 'I need to know', or 'I cannot do that' and 'I want to be able to do that'.

Isolated models. Models of a new piece of knowledge come into mind gradually and have a long-term perspective. For instance, the concepts of place value, negative number, or straight line develop over many years at a preparatory level. For more complex knowledge, the stage of isolated models can be divided into four sub-stages as described in (Hejný, 2011a, 2011b).

This stage ends with the creation of the community of isolated models. In the future, other isolated models will come to a pupil's mind, but they will not influence the birth of the generic model.

Generic models. In figure 1, the generic model is placed over the isolated models indicating its greater universality. The generic model is created from the community of its isolated models and has two basic relationships to this community:

1. it denotes both the *core* of this *community* and the core of *relationships* between individual models, and
2. it is an *example* or *representative* of all its isolated models.

The first relationship denotes the construction of the generic model; the second denotes the way the model works. The absence of a generic model leads to mechanical knowledge. For example, the pupil who creates a generic model of the formula for the area of a triangle by solving a series of problems understands the formula. The pupil who takes the formula over by transmission has mechanical knowledge only. He/she can apply it on standard problems, however, when he/she forgets the formula there is no way for him/her to rediscover it.

2.3 Scheme

This term will be first illustrated by an everyday situation. When someone asks you about the number of doors, windows or lamps in your flat or house, you will probably not be able to give an immediate answer. However, in a little while you will answer the question with absolute certainty. You will imagine yourself walking from one room to another and counting the objects in question. Both the required pieces of information and many other data about your dwelling are embedded in your consciousness, as a part of the scheme of your flat. We use schemes to recognize not only our dwellings but also our village, our relatives, interpersonal relationships at our workplace, etc.

From the point of view of mathematics education the key question is: How did the scheme of our flat enter our consciousness? How was it created? The answer is obvious. This scheme is not the result of learning the curricular topic "The furnishing of our flat" in September, "Lamps and carpets" in October, "The Kitchen" in November etc. at school. The scheme is an outcome of our everyday experience in the given environment. In some activities our attention focuses on some part of the flat (we hang up a picture, clean windows, move furniture, tidy up, …) but all of these particulars are perceived as parts of one whole.

This implies that efficient mathematics education should be adapted to the natural process through which we learn in real life: various mathematical environments should be created and the pupil should be allowed to move in them, i.e., to solve a variety of problems in the environment. Our experience with scheme-oriented education clearly shows that a well-developed environment that allows pupils to move freely in it results in 1) substantial improvement of the pupils' attitude to mathematics, and 2) markedly better mathematical knowledge.

The notion of scheme was elaborated in cognitive psychology by Anderson (1983) and we take it over in a way described by Gerrig (1991, pp. 244–245):

> "Theorists have coined the term schemes to refer to the memory structure that incorporate clusters of information relevant to comprehension … A primary insight to scheme theories is that we do not simply have isolated facts in memory. Information is gathered together in meaningful functional units."

The notion of a substantial learning environment was introduced in mathematics education by Wittmann (2001) who stressed its basic property: it enables us to formulate a series of problems which help a pupil to understand deep ideas of mathematics. The concept of a deep mathematical idea is elaborated in Semadeni (2002). In our approach, three additional requirements are placed on a mathematical learning environment: connection to a pupil's life experience, long-term nature (it is usable for pupils of different ages, at best from grade 1 to grade 12) and differentiated nature (it enables to pose problems to cater for needs of individual pupils). Wittman's idea of substantial learning environments has found followers in the Czech Republic, too (Jirotková, 2004; Hejný, Jirotková, & Kratochvílová, 2006; Hejný & Slezáková, 2007; Hejný & Jirotková, 2004, 2007, 2009a, 2009b, 2010; Hošpesová et al., 2010; Tichá & Hošpesová, 2011; Jirotková & Marchini, 2011).

2.4 Scheme-oriented education

Scheme-oriented education is based on the construction of different schemes that interlink, combine and form a dynamic network of a pupil's mathematical knowledge and skills. For example:

– *area* starts with generic models of the area of square, rectangle, triangle, … within the environments 'tessellation', 'paper folding', 'grid paper', 'geoboard' and 'stick shapes';

– *small natural number* starts with generic models of address, status, operator of change and operator of comparison within environments 'stepping', 'money', 'pebbles', 'rhymes', 'ladder', …;

– *fraction* starts with generic models of 'one half', 'one quarter', 'divide into halves', 'equal sharing', … within environments 'pizza', 'stick' and 'chocolate'.

The traditional transmissive teaching of mathematics is based on a teacher's exposition. The teacher introduces pupils to concepts, relationships, processes and situations and leads them towards remembering definitions and formulas and imitations of procedures and algorithms. As a result of this educational style, most pupils cannot discover mathematics, their creative activity is limited and their insight into mathematics suffers from a lack of understanding of the subject matter.

For scheme-oriented education, it is necessary that a pupil is intellectually autonomous in the sense that he/she discovers new ideas or gets to them by communicating with classmates or takes them over from the classmates. It follows that a teacher's role is indispensable for scheme-oriented education and now we consider this.

3 Realisation of Cognitive Goals in the Teaching of Mathematics

A teacher is a key agent in the realisation of the above cognitive goals. It is the teacher who presents pupils with adequate problems and organises their discussion in such a way that it proceeds to the required knowledge.

3.1 Scheme-oriented vs. transmission educational style

A difference between the two educational styles can be best illustrated by the following simulated dialogue.

Story A

Two 5th grade teachers, Alena and Anezka, are speaking about their experience with the introduction of a divisibility test by 9.

> Alena: "Already in grade 3, when pupils learn to divide, they get dozens of tasks of the type *What can digit X be so that the number 4X2 is divisible by nine?* Then in grade 4, they solve a task again, for example, *Find a three digit number XYZ divisible by nine*

so that X + Y + Z = 8. Pupils find out that such a number does not exist but if *X + Y + Z = 9*, then there are a lot of such numbers. At this point, some pupils formulate the divisibility test by 9, so far only for three digit numbers. And only now in grade 5 when a half of the class discover the rule for three digit numbers, some pupils discover the same rule for four and five digit numbers. At the end, they will state a general rule and justify it with the help of a calculator for numbers with up to 10 digits."

Anezka: "Your way is interesting but far too long. First for three digit numbers, then for four digits, then five digits – it requires a lot of time. Moreover, I doubt that weaker pupils understand what it is about. They do not have time to practise the rule sufficiently. Of course, for two or three best pupils, it is inspiring, but most pupils, at least in my class, would require a clear rule which they acquire easily. By the way, you speak about dozens of problems, do you have a collection of such problems?"

When asked, Alena is able to show dozens of tasks which pupils solved in grade 3 and 4 and for some, she also shows erroneous pupils' hypotheses and examples from the follow up discussions in the class.

Anezka's last sentence shows that mathematics educators have an important task: to create sets of sufficiently varied (in terms of content and difficulty) problems to topics of school mathematics which would enable pupils to discover generic models via isolated models. This can easily be done by didactic mathematical environments (Hejný, 2011b).

3.2 Teacher's role

The teacher's work within the scheme-oriented education is guided by the following principles.

The teacher should

1. create optimal climate for learning: no pupil is frustrated, no pupil is bored, the teacher shares the pupils' successes, the teacher encourages pupils who might give up mathematics, the teacher builds the pupils' self esteem;

2. leave pupils space for their considerations: does not impose his procedures in their minds even when the pupils' ones are clumsy, does not orient pupils towards a quick solution by overly leading questions, does not interrupt pupils' thinking processes, when a pupil asks a direct mathematical question, the teacher values the question "It is an interesting question" and asks the class to look for answers;

3. lead pupils towards discussions: when hosting the discussion he/she makes space for erroneous ideas and weaker pupils. For example, when two pupils discover two different correct algorithms for written addition, the teacher lets each one choose which one he/she will use;

4. not point out mistakes to pupils: let pupils discover the mistakes, or provides them with a suitable task where they can see them; the mistake is understood as a way towards a deeper understanding of the investigated situation, he/she teaches pupils to analyse mistakes, mainly by analysing their own mistakes;

5. provide pupils with adequate tasks: each pupil, both of a high and low ability, solves the task which corresponds to his/her abilities and thus can experience

joy from success; problems assigned which do not allow for differentiation are frustrating for weaker pupils and boring for high ability pupils. On the other hand, the tasks which allow for both "quick" and "slow" solving strategies are suitable (see story C);

6. lead pupils (by his/her own approach to mathematics) to the need to understand not only mathematics as such but also the way it might be understood by their classmates: when the problem is solved by various solving strategies, the pupils broaden not only their horizons in mathematics, but also their understanding of other people's thinking processes and opinions; in this sense, mathematics contributes to critical thinking and cultivates pupils' democratic awareness.

3.3 Illustration

The illustration describes the first 9 minutes of a video recorded in grade 3 elementary class in February 2012. There were 7 boys and 14 girls in the class. The teacher J. M. has been teaching the pupils since September 2011. The video recording was made by a student A. Sukniak who happened to observe the lesson.

Story B

The teacher set the task: *Mother bought 5 lollypops, 3 CZK each, and chocolate for 12 CZK. How much the purchase cost?* Soon, the pupils lifted scratch writing tablets with their solutions. The teacher observed the tablets and after one minute, she wrote three results on the blackboard without the evaluation of their correctness: 21, 22, 27. Then she asked Lucka to explain the result of 21. When explaining, the girl herself found out that she made a mistake in 3×5 equals 9 and corrected the result into 27. Then the teacher without reacting to the approving voices from the class asked Lukas to explain his result of 22. The boy immediately said that it was not correct and corrected his result. Three minutes passed from setting the task.

The teacher asked the pupils to write their strategy on the blackboard. Three recordings appeared in two minutes: the first $5 \cdot 3 + 12 = 27$, the second $5 \cdot 3 = 15$, $15 + 12 = 27$, Lukas's, $5 + 5 + 5 = 15 + 12 = 27$. There were only approving commentaries to the first two: "Yes, I have the same." "It should be written one below each other." Some protests could be heard for the last record. The teacher asked Jolana: "What do you not like about it?" Jolana said that $5 + 5 + 5 = 15 + 12$ was not correct. The teacher asked Lukas if he understood Jolana. Lukas stated that not at all. Then Kristyna made the second attempt to explain the mistake, in a clear and persuasive way: "Five plus five plus five is fifteen; it is true up to here; when you add twelve, it is no longer true." However, Lukas did not understand this explanation either. The teacher asked the class who would explain it to Lukas. Four girls put their hands up and the teacher called out Misa: "In fact, he had one problem. He had it in one problem, but when Lucka puts it into two problems, then it is no longer the one problem and it does not hold that twelve plus fifteen equals fifteen." Lukas: "I do not understand this one at all." Misa took Lukas to the blackboard, covered

50   number 27 with her hand and read backwards what Lukas had written: "Twelve plus fifteen is five plus five plus five, it is true?" After some hesitation Lukas said: "No." Misa: "So why did you write it?" Lukas laughed and went to his desk. Nine minutes passed. The end of the story.

The story illustrates five of the six principles above.

1. No pupil was unnecessarily frustrated, the teacher was not nervous and did not press Lucka who spoke quite slowly. As far as we can say from the observation and the video recording, no one was bored, not even quick Misa. Misa, as transpired later, was solving a meta-cognitive problem of Lucka's and Lukas's mistake.

2. The teacher let each pupil explain his/her strategy. She gave them space for their considerations in which she did not intervene. She let the pupils look for the explanation for Lukas and did not enter the discussion even after the first two unsuccessful attempts. It is obvious that after common four month work, a didactic contract (Brousseau 1997; Sarrazy, Novotná 2005) between her and the class had set up in which the pupil addressed by the teacher did not feel fear but an opportunity to express himself/herself.

3. The teacher first gave precedence to two erroneous results and only after the explanation of mistakes she asked pupils to write strategies on the board. This encouraged a 6 minute discussion which the teacher only moderated. When observing their classmates, the pupils acquired experience with other solving strategies.

4. The teacher did not point to any of the three mistakes which appeared. The pupils identified them and removed them. The boy with the result of 21 and Lukas realised the cause of the mistake, too. None in the class found out the inaccuracy in Lukas's record. The price of five lollypops 3 CZK each should have been written $3 + 3 + 3 + 3 + 3$ and not $5 + 5 + 5$. The teacher did not point to it because she wanted to keep the dynamics of the lesson.

5. The fifth principle could not be seen in story B because the task used was an introductory one, presented to the whole class.

6. From frequent comments from the class to the records on the board, it is clear that the pupils followed their classmates' work with more attention than is customary in grade 3. The discussion among pupils bore evidence of the fact that their attitude to mathematics is creative and not consumerist. An excellent example of only four month impact of the teacher was Misa's precise analysis of Lukas's mistake and effective didactic mastering of the situation. First, Misa formulated the cause of Lukas's incomprehension of his mistake in a not very comprehensible manner, rather to herself and the teacher, and then she demonstrated it in a short but effective dialogue with Lukas. The core of Lukas's conviction that his strategy was correct lies in a procedural understanding of the calculation (described in literature many times). He did not understand an argument that he had to write it in a conceptual way. He realised that the conceptual record was correct, but he did not know what was wrong about his record which exactly copied the course of ideas. By reading Lukas's procedure backwards, Misa conceptualised the solution and Lukas understood his mistake.

The fifth principle is illustrated by subsequent story C which comes from our archive.

Story C

Grade 3 pupils were solving the task which was accompanied by figure 2.

Four sticks are needed for a square. How many sticks do we need to create windows of two squares? How many sticks to make a window of three squares? Of four squares? Of five squares?
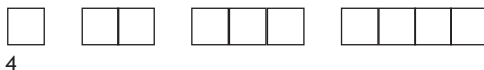

Figure 2 How many sticks?

Each pupil had enough sticks available. A similar problem was solved by the pupils a month ago when the windows were triangular.

In the first 6–7 minutes, Verka created the first two figures on the desk, she redrew them to her exercise books and wrote number 4 and 7 to them. Gusta did the same thing. Richard had already made the third shape, was redrawing it to his exercise book and whispered "ten, ten, …" not to forget that he would need 10 sticks. Most pupils proceeded further in their work. Hanka had already solved the task. She proceeded in such a way that she added another square to the created windows of two squares and counted the number of sticks. Next she added the fourth square and counted the number of sticks again. At the end, she proceeded similarly for the fifth square. She only wrote numbers 4, 7, 10, 13 and 16 in her exercise book and ran to show it to the teacher. Some other pupils went to the teacher, too. He asked them to mutually compare their answers. Those who had finished were encouraged to count sticks for longer windows. Not all the pupils had the same results, so some corrected their work. The pupils who were sure of their results were counting other windows.

Then suddenly, Tomas cried out: "It is always plus three." He discovered that for the sequence of found numbers it holds: the subsequent number is the previous one plus three. More than a minute before that, Imrich whispered the rule to the teacher. The teacher did not comment on its correctness and asked him to find out how many sticks he would need for windows of ten, twenty, thirty, … a hundred squares. (The teacher knew that Tomas safely counted to 1000.)

The teacher addressed the class and asked them if they understood what Tomas was saying. Karla and Danka said that they found it, too, but Veronika said that she could not understand. Tomas wrote a sequence of results 4, 7, 10, 13, 16 on the blackboard and showed that his rule was valid. Several pupils started to check if we needed 19 sticks for a window of six squares. Radka confirmed that Tomas was right, that she "had already counted it". Verka and Richard did not participate in the discussion, went on with their work and were just finishing the last window. After a while, Imrich said to the teacher that for the tenth window, 31 sticks are

needed, for the twentieth one 61 sticks, for the thirtieth one 91 sticks and for the hundredth window 301 sticks. Let us add that the next day Imrich came with the rule: The number of sticks is three times of the number of squares in the window plus one. After a month, when most of the pupils knew Imrich's rule, Hanka came with the idea connected to the earlier used triangular windows. She found out that the number of sticks was twice the number of triangles plus one. The end of the story.

Story C shows that a suitably chosen task can be adequate for both mathematically weaker pupils such as Verka and Richard and for the mathematically strongest pupil Imrich and for all the others. Verka and Richard made five isolated models and were able to continue with looking for others independently. After solving the given problem, most of the pupils were able to continue with its extension. Tomas discovered a procedural generic model in the sequence of sticks. Some other pupils made the same discovery and others accepted it immediately because they were ready for it by their own activity. Imrich, who discovered this generic model as the first pupil, was assigned a task from the teacher which led him to the conceptual generic model. From this point of view, the task was chosen appropriately as numbers 31, 61 and mainly number 301 meaningfully suggested how the number of sticks relates to the number of squares. It is quite natural that different pupils reached different levels in understanding the relationship between the number of squares and the number of sticks. Only after some other pupils discovered the conceptual generic model in a few days, did the teacher ask one pupil to show the rule to the whole class. It can happen that in such a case even the pupil who only took over the discovery makes his/her own discovery at this point. We could see that in Hanka's case.

4 Conclusions

We have shown that the teaching style aimed at the pupil's intellectual development in mathematics can be realised by scheme-oriented teaching. We have justified the key role of generic models for building schemes. We have illustrated the educational strategy as a process which leads to the birth of a generic model through isolated models. We assert that the absence of a generic model leads to mechanical knowledge and the development of a pupil's creativity is determined by the process of looking for generic models. In more than ten years of research we have been oriented towards both pupils and their teachers. In this study, we have looked deeply into the diagnosis of the teacher's educational style. However, much more is needed, namely a tool which can be used for the description, analysis and comparison of teachers' teaching styles. Such a tool together with its development and application is presented in Jirotková (2012) in this issue.

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