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Reciprocal Equilibria in Link Formation Games

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Abstract We study non-cooperative link formation games in which players have to decide how much to invest in relationships with other players. A link between two players is formed, if and only if both make a positive investment. The cost of forming a link can be interpreted as the value of privacy. We analyze the existence of pure strategy equilibria and the resulting network structures with tractable specifications of utility functions. Sufficient conditions for the existence of reciprocal equilibria are given and the corresponding network structure is analyzed. Pareto optimal and strongly stable network structures are studied. It turns out that such networks are often complete.

Keywords Link formation games, reciprocal equilibrium, complete network
JEL classification C72, D43

1. Introduction

We study non-cooperative link formation games in which players have to decide how much to invest in relationships with other players. A link between two players is formed, if and only if both make a positive investment. The value of a link depends on the size of investments, and this value can be different for different players. The cost of forming a link can be interpreted as the value of privacy, or the opportunity cost of lost privacy.

Friendships, partnerships, and researchers' collaboration networks are prime examples of situations that could be modeled this way. Friendships could be strong or weak and two people in a relationship could value it differently. Researchers may spend different levels of effort in their joint projects, and they could value their cooperation differently. It is therefore important to understand what kind of factors affect agents' choices in such situations, and how the equilibrium network looks like.

We analyze the existence of pure strategy equilibria and the resulting network structures with tractable specifications of utility functions. Sufficient conditions for the existence of reciprocal equilibria are given and the corresponding network structure is analyzed. Pareto optimal and strongly stable network structures are studied. It turns out that such networks are often complete.

Each player has a fixed amount of a single resource like time or effort that he can invest in relationships with other players and/or use for his own private benefit. The more two players invest in their mutual relationship, the higher is the utility to both

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players from this relationship. Since resources are limited, utility from privacy or from other relationships decreases, and there is a tradeoff between relationships. Decisions are made simultaneously and pure strategy Nash equilibria are searched for.

We show that a reciprocal equilibrium with a complete network (or a network with complete components) exists in many symmetric or anonymous link formation games (Theorems 1 and 2). In such an equilibrium players i and j invest equal amounts in their mutual relationship. Bauman (2015) studies reciprocity of equilibria in symmetric games with strictly concave valuations of privacy and constant returns to scale Cobb-Douglas utilities from relationships.

Network structure in a reciprocal equilibrium depends on players' valuations of privacy. If these valuations are linear functions, then reciprocal equilibria often exhibit *homophily* (Theorem 3): links are more likely to be formed between similar players (Currarini et al. 2009).

Equilibria with a complete network exists under variety of circumstances when reciprocity is not demanded, for example in *semi-symmetric* games with bilateral strategic complements or substitutes (Theorems 4 and 5). In semi-symmetric link formation games players have common preferences over other players as friends. In such cases it is important to understand how the popularity or status of an agent affects his behavior in the network. Salonen (2015) studies the relation between popularity and some well-known network centrality measures in semi-symmetric games.

In the class of models studied in this paper, Pareto optimality of a network structure implies in many cases that network must be complete (Propositon 1). Similarly, strongly stable equilibria (Bloch and Dutta 2009) have often complete networks as well (Proposition 2). It is shown at the end of Section 4.1. that *any* Pareto optimal or strongly pairwise stable equilibrium must have a complete network, when utilities have a strictly concave Cobb-Douglas form.

Completeness of a network sounds rather extreme if the player set is very large. A more moderate interpretation of these results would be that networks consist of completely connected components. Be this as it may, Bloch and Dutta (2009) get results that efficient or strongly stable networks are stars. It is therefore necessary to compare the underlying assumptions of our models.

We assume that players get utility only from private consumption or direct links (relationships) with other players, and that a relationship of two players gives positive utility only if both players have made a positive investment. Bloch and Dutta (2009) assume that players get utility also from indirect connections, i.e. from friends of friends, and that a link between two players is formed even if only one of the players has made a positive investment. In our model two linked players may value the relationship differently, whereas in their model the values are identical.

The model of Bloch and Dutta (2009) may be more natural in situations where links have instrumental value, like communication networks. Since direct links are not absolutely necessary for information transmission, complete networks need not be efficient structures. Our model is perhaps better suited in cases where links have intrinsic value, like friendships. In such cases indirect connections may be very poor substitutes for direct links, and increasing the number of direct links becomes both individually and socially optimal.

There is a large literature of link formation games where the link strength can take only two values: either it is 1 (link is formed) or 0 (link is not formed). Jackson and Wolinsky (1996) is the seminal paper of this strand of literature (see Jackson and Zenou 2015 for a comprehensive review of network games). Cabrales et al. (2011) analyze a linear quadratic game with productive investments and link formation where link strengths can be nonnegative real numbers. Rather than choosing each link intensity separately, a player chooses one real number that describes his socialization effort. Strengths of individual links are then determined jointly, given socialization efforts of all players. The resulting network determines the profitability of productive investments.

In our model players invest in each link separately, and the utility from equal investments in different links may be different. So the links of a player may represent very different relationships with other players, although seemingly a player decides only how to share a homogeneous resource among his friends.

The paper is organized in the following way. The notation is introduced in Section 2. In Section 3 some simple models with Cobb-Douglas functions are analyzed. Main results are stated in Section 4.

2. The Model

A tuple $W = (N,g)$ denotes an unweighted, undirected *network* with a finite node set N and a link set g. The link set g specifies which nodes $i, j \in N$ are directly connected. Such a link may be denoted by $ij \in g$ with the understanding that $ji = ij$. In this paper loops are ignored so $i \neq j$ if i and j are linked. If it is clear what the node set is we may denote a network simply by g.

Given a network $W = (N,g)$ and $i, j \in N$, there exists a *path* between i and j, if there exists nodes $i_0,\dots,i_K$ such that (i) $i_0 = i, i_K = j$; (ii) $i_k i_{k+1} \in g$ for all $k = 0,\dots,K-1$; (iii) all nodes are distinct except possibly i_0 and i_K. A network $W = (N,g)$ is *connected* if there exists a path between any two nodes $i, j \in N$.

A subset $A \subset N$ is a *component* of a network $W = (N,g)$, if (i) there exists a path between any two nodes $i, j \in A$; (ii) there are no links between A and $A^c \equiv N \setminus A$. A node set N can always be partitioned into connected components. A network $W = (N,g)$ is connected, if N is a component. A component A is complete, if for every $i, j \in A$ there is a link $ij \in g$. A network $W = (N,g)$ is complete, if N is a complete component.

A *normal form game* $G = \big(N, (S_i)_{i \in N}, (u_i)_{i \in N}\big)$ specifies a player set N, a set of pure strategies S_i and a utility function $u_i : S \longrightarrow \mathbb{R}$ for each player $i \in N$, where $S = \Pi_i S_i$, the product of strategy sets, is the set of *strategy profiles*.

A game G is *symmetric*, if $S_i = S_j$ for all $i, j \in N$, and $u_i(s) = u_j(s')$ for all $i, j \in N$, for all $s, s' \in S$ such that $s_i = s'_j, s_j = s'_i$ and $s_k = s'_k$ for all $k \neq i, j$.

A game G is *anonymous*, $S_i = S_j$ for all $i, j \in N$, and $u_i(s) = u_i(s')$ if the only difference between s and s' is that $s_j = s'_k$ and $s_k = s'_j$ for some $j, k \neq i$.

Given $s \in S$, we may denote $s = (s_i, s_{-i})$ when we want to emphasize that i chooses

s_i. A pure strategy Nash equilibrium is a strategy profile $s \in S$ such that

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall i \in N, \forall s_i' \in S_i. \tag{1}$$

Given a symmetric game G, a strategy profile s is a *symmetric equilibrium*, if $s_i = s_j$ for all $i, j \in N$.

We study *link formation games* of the following type. The set of pure strategies of player $i \in N$ is

$$S_i = \left\{ s_i \in R_+^N \mid \sum_j s_{ij} = 1 \right\}.$$

An interpretation is that each player i has one unit of time or effort to be shared with other player j including i himself. The utility function of player i is

$$u_i(s) = \sum_{j \neq i} U_{ij}(s_{ij}, s_{ji}) + V_i(s_{ii}), \tag{2}$$

where $U_{ij} : [0,1]^2 \longrightarrow \mathbb{R}_+$ is a function giving the utility for player i from investments s_{ij}, s_{ji}. The function U_{ij} has the following properties: (i) $U_{ij}(0, s_{ji}) = 0 = U_i(s_{ij}, 0)$ for all s_{ij}, s_{ji}; (ii) U_{ij} is strictly concave and differentiable in s_{ij} for any given $s_{ji} > 0$; (iii) U_{ij} is strictly increasing and continuous on $(0,1] \times (0,1]$.

The function $V_i : [0,1] \longrightarrow \mathbb{R}_+$ tells how much player i values privacy. The function V_i is concave, strictly increasing, differentiable on $(0,1)$, and $V_i(0) = 0$.

In anonymous link formation games $U_{ij} = U_i$ for all $i, j \in N$, and hence $u_i = U_i + V_i$. In symmetric link formation games $u_i = U + V$ for all $i \in N$. [In a link formation game the identity of strategies $s_i = s_j$ is understood so that $s_{ii} = s_{jj}$, $s_{ij} = s_{ji}$, and $s_{ik} = s_{jk}$ for all $k \neq i, j$.]

We say that a link formation game $G = \big(N, (S_i)_{i \in N}, (u_i)_{i \in N}\big)$ is *semi-symmetric*, if there are functions U and V such that

$$u_i(s) = \sum_{j \neq i} p_j U(s_{ij}, s_{ji}) + c_i V(s_{ii}), \forall s \in S, \tag{3}$$

for some parameters $p_j > 0, c_i > 0$, for all $i, j \in N$. So there is a common ordering of players such that player j is considered as a more valuable friend than i, if $p_j > p_i$. The cost parameters c_i reflecting the value of privacy could be player specific.

There are two different interpretations of the network model associated with the game. We may think that the network is undirected and unweighted, and the investments describe the intensity of the relationship, or how much the agents utilize a given link and how much they benefit from it. Alternatively, investment s_{ij} gives the strength of the link from i to j, and players get nonzero utility from a relationship only if both make a positive investment. In this case the network is directed and weighted. To keep notation simple, we gave formal definitions for the undirected unweighted network only.

Next we give some definitions that are needed in the main theorems.

Definition 1 (Bilateral strategic complements). U_{ij} is twice continuously differentiable on $(0,1) \times (0,1)$ with $\partial^2 U_{ij} / \partial s_{ji} \partial s_{ij} > 0$, $i \neq j$.

Bilateral strategic complements imply $\partial^2 u_i/\partial s_{ji}\partial s_{ij} = \partial^2 U_{ij}/\partial s_{ji}\partial s_{ij}$ by (2). Since $s_{ii} = 1 - \sum_{j\neq i} s_{ij}$ the usual strategic complements condition is not satisfied: if s_{ji} increases, then s_{ij} increases but s_{ik} decreases for some $k \neq j$ when i uses a best reply.

Analogously, bilateral strategic substitutes mean $\partial^2 U_{ij}/\partial s_{ji}\partial s_{ij} < 0$ holds on $(0,1)\times(0,1)$, for all players i.

Definition 2 (**Increasing derivative on the diagonal**)**.** A function $U_{ij} : [0,1]^2 \longrightarrow \mathbb{R}_+$ has (strictly) increasing derivative on the diagonal, if

$$\frac{\partial U_{ij}(y,y)}{\partial x_1}(<) \leq \frac{\partial U_{ij}(z,z)}{\partial x_1}, \text{ for all } y < z.$$

If both i and j invest y in their relationship, the marginal utility for i increases in y. If the inequality in Definition 2 is reversed, we say that U_{ij} has (strictly) decreasing derivative on the diagonal. If equality holds for all $y < z$ we say that U_{ij} has constant derivative on the diagonal.

Note that if U_{ij} is (jointly) concave, then it has a decreasing derivative on the diagonal. On the other hand the Cobb-Douglas function $f(x,y) = x^a y^b$ is concave in both arguments separately and has increasing derivative on the diagonal, if $0 < a,b < 1$, and $a+b \geq 1$. If U_{ij} is homogeneous of degree $\alpha \geq 1$ ($0 < \alpha \leq 1$), then U_{ij} has increasing (decreasing) derivative on the diagonal. Homogeneity is clearly a much stronger assumption than increasing and decreasing derivative conditions.

If a game is not symmetric, a symmetric equilibrium need not exist. However, behavior may be nearly symmetric also in non-symmetric games. The following is a pairwise or bilateral symmetry condition that seems natural in the context of friendship networks.

Definition 3 (**Reciprocal equilibrium**)**.** An equilibrium s of a link formation game $G = \big(N,(S_i)_{i\in N},(u_i)_{i\in N}\big)$ is *reciprocal*, if $s_{ij} = s_{ji}$ for any pair $i,j \in N, i \neq j$ of players.

An *interior equilibrium* s of a link formation game is such that $s_{ii}, s_{ij} > 0$ for all players i,j. The network corresponding to an interior equilibrium is complete. Note that if $s_{ii} = 0$ for some player i, then s cannot be an interior equilibrium. If $s_{ij} > 0$ for all i and for all $j \neq i$, then the network is complete.

3. Examples

Let us first analyze some simple examples based on Cobb-Douglas functions U_{ij}.

Example 1. Let $G = \big(N,(S_i)_{i\in N},(u_i)_{i\in N}\big)$ be a semi-symmetric game with bilateral strategic complements such that

$$u_i(s) = \sum_{j\neq i} p_j s_{ij}^{\alpha} s_{ji}^{1-\alpha} + c_i\big(1 - \sum_{j\neq i} s_{ij}\big),$$

where $0 < \alpha < 1$, and $c_i, p_j > 0$. For generic values of parameters α, c_i, p_j all equilibria s satisfying $s_{ii} > 0$ for all i are autarkic. That is, $s_{ii} = 1, \forall i$. To see this, suppose that in equilibrium all the values s_{ij}, s_{ji}, s_{ii} and s_{jj} are strictly positive for some players i, j. Then the corresponding first order conditions for players i and j satisfy:

$$\begin{aligned} \alpha p_j s_{ij}^{\alpha-1} s_{ji}^{1-\alpha} &= c_i \\ \alpha p_i s_{ji}^{\alpha-1} s_{ij}^{1-\alpha} &= c_j \end{aligned} \tag{4}$$

These equations imply

$$\alpha^2 p_i p_j = c_i c_j, \tag{5}$$

which does not hold for generic values of α, c_i, p_j. Namely, let $\mathbb{R}^{2n+1}_{++}$ be the set of all strictly positive vectors $x = (\alpha, p_1, c_1, \ldots, p_n, c_n)$. Take any two players i, j. The subset of vectors $x \in \mathbb{R}^{2n+1}_{++}$ such that $\alpha p_i p_j = c_i c_j$ is closed and has an empty interior in $\mathbb{R}^{2n+1}_{++}$. Since there are only finitely many players, the subset B such that equation (5) does *not* hold for any two players i, j is such that the closure of B contains $\mathbb{R}^{2n+1}_{++}$. Hence, generically $\alpha p_i p_j = c_i c_j$ does not hold.

Suppose that for each pair p_t, c_t there is a group N_t of players with these parameters in their utility functions. Then the genericity result above implies that typically links are formed only within each group N_t, if $s_{ii} > 0$ for all players. In this case equilibria exhibit *homophily*: links are formed only between similar players (Currarini et al. 2009). Note however that other kinds of equilibria may exist if $s_{ii} = 0$ for some players.

The game G of Example 1 has constant derivative on the diagonal and a linear V_i function. Theorem 2 below shows that if G is an anonymous game and V_i functions are strictly concave, then an interior *reciprocal* equilibrium often exists.

Let us modify the game G of Example 1 slightly so that interior equilibria exist.

Example 2. Let $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ be a game such that

$$u_i(s) = \sum_{j\neq i} p_j s_{ij}^{\alpha} s_{ji}^{\beta} + c_i\big(1 - \sum_{j\neq i} s_{ij}\big),$$

where $0 < \alpha, \beta$, $\alpha + \beta < 1$, and $c_i, p_j > 0$, for all $i, j \in N$. Let $p_{ij} = p_j / c_i$, and the first order conditions for an interior equilibrium for players i, j are:

$$\alpha p_{ij} s_{ij}^{\alpha-1} s_{ji}^{\beta} = 1 \tag{6}$$

$$\alpha p_{ji} s_{ji}^{\alpha-1} s_{ij}^{\beta} = 1 \tag{7}$$

Solving for s_{ij} gives us

$$s_{ij} = \alpha^{1/[1-\alpha-\beta]} \left[p_{ij}^{1-\alpha} p_{ji}^{\beta}\right]^{1/[(1-\alpha)^2-\beta^2]}, \forall i, j \in N. \tag{8}$$

Note that s_{ij} is an increasing function of both p_{ij} and p_{ji}. If c_j increases, the value of privacy for j increases, and p_{ji} decreases. Then j invests less in his relations with other agents. Consequently, also s_{ij} decreases by bilateral complementarity.

If all players are identical, then $p_{ij} = p/c$ for all i, j, for some p, c. A symmetric interior equilibrium exists if

$$\alpha p < c\left[\frac{1}{n-1}\right]^{1-\alpha-\beta}.$$

As n increases, this inequality holds if p decreases or c increases sufficiently. This holds since in symmetric equilibrium marginal utility from links increases as n increases because $\alpha+\beta<1$. At an interior equilibrium $s_{ii}>0$, and therefore the value of privacy c must increase relative to p.

For a nonsymmetric example, let $n=11, \alpha=1/4, \beta=1/2$, and $p_1=p, p_2=p^2,\ldots,p_n=p^n$ for some $p\in(0,1)$. If $c_i=1$ for all i, then an equilibrium with a complete network is given by

$$s_{ij}=4^{-4}\left[p^{2i+3j}\right]^{4/5}, \tag{9}$$

from which we can compute that

$$s_{ji}=\left[p^{i-j}\right]^{4/5}s_{ij}, \text{ for } j<i.$$

The players who are highly ranked by the society (low i and high p_i) invest less in relationships than lower ranked players. Take $i=4$ and $j=3$. Then $s_{34}=p^{4/5}s_{43}$, and therefore $s_{34}<s_{43}$.

For another numerical example, assume $p_j=1$ for all players j, and $c_1=c, c_2=2c,\ldots,c_n=nc$, for some $c>1/2$, and let the other parameters have the same values as above. Then the following values characterize an equilibrium with a complete network:

$$s_{ij}=4^{-4}\left[i^{-3}j^{-2}c^{-5}\right]^{4/5}, \tag{10}$$

from which we can compute that

$$s_{ji}=\left[\frac{i}{j}\right]^{4/5}s_{ij}.$$

The players with high value of privacy (high c and i) invest less in relationships than players with a low value of privacy. Take $i=4$ and $j=3$. Then $s_{34}=(4/3)^{4/5}s_{43}$, and therefore $s_{34}>s_{43}$.

4. Results

The existence of equilibria is not a problem in our model, since a strategy profile s such that $s_{ii}=1$ and $s_{ik}=0, k\neq i$, for all $i\in N$ is trivially an (autarkic) equilibrium, and also a reciprocal equilibrium. Here is a more interesting existence result for symmetric games. All long proofs are relegated in the Appendix.

Theorem 1. *A symmetric link formation game* $G=\left(N,(S_i)_{i\in N},(u_i)_{i\in N}\right)$ *has a non-trivial reciprocal equilibrium with complete components, if and only if there exists*

$x \in (0,1)$ *such that*

$$\frac{\partial U(x,x)}{\partial x_1} - V'(1-x) \geq 0. \tag{11}$$

Proof. See Appendix.

The condition (11) says it is better to form one reciprocal link than not to form any links with other players.

Reciprocal equilibria may exist also in nonsymmetric games.

Theorem 2. *Let $G = \big(N,(S_i)_{i\in N},(u_i)_{i\in N}\big)$ be an anonymous link formation game with the following properties: (i) constant derivative on the diagonal, (ii) V_i is strictly concave. Assume also that if all players $i \in N$ have the same utility function u_i, then the corresponding symmetric game would have a symmetric equilibrium s^i such that the resulting network is complete, for any $i \in N$. Then there exists a reciprocal equilibrium such that the resulting network is complete.*

Sketch of a proof. Theorem 1 gives necessary and sufficient conditions for the existence of a reciprocal equilibrium in symmetric games. The idea of the proof is the following. Formulate a symmetric game corresponding to each of the utility functions u_i that players in a (nonsymmetric) game G have. By assumption, each of these games has a symmetric equilibrium with a complete network. A reciprocal equilibrium for G can be recursively constructed from these symmetric equilibria. For details see Appendix. □

Note that if V_i is linear, then Theorem 2 may not hold by Example 1. Theorem fails if V_i is linear even if U_i is assumed to be strictly concave as the following result demonstrates. We say that a node subset C is a *clique*, if there is a link between every two nodes $i, j \in C$.

Theorem 3. *Suppose $G = \big(N,(S_i)_{i\in N},(u_i)_{i\in N}\big)$ is an anonymous link formation game such that (i) derivative is strictly decreasing on the diagonal, and (ii) V_i is linear and $U_i = U$. If there is a reciprocal interior equilibrium s such that the equilibrium network has a clique C, then players $i \in C$ have the same utility functions $u_i = U + V_i$.*

Proof. By condition (ii), $V_i(s_{ii}) = c_i s_{ii}$ for some constant $c_i > 0$. For each $i \in C$, there is at most one x^i such that $\partial U_i(x^i,x^i)/\partial x_1 = c_i$ by condition (i). For $i \in C$ this equality must hold in the reciprocal equilibrium s since $s_{ii} > 0$. If $c_i \neq c_j$, then $x^i \neq x^j$ because $U_i = U_j$. Therefore if C is a clique in an equilibrium network and $i, j \in C$, then $c_i = c_j$ and hence players in C have the same utility functions. □

Remark 1. Note that Theorem 3 holds also if condition (i) is replaced by the condition that derivative is strictly increasing on the diagonal. Of course, marginal utility from link formation may be so large as compared to the cost parameters c_i, that $s_{ii} = 0$ in equilibrium. Then there could exist reciprocal equilibria with a complete network even if players have different cost parameters c_i.

We show next that if a game has bilateral strategic complements, then with the same or slightly weaker assumptions as in Theorem 3 there exists an equilibrium such that the equilibrium network is complete. By Theorem 3 this equilibrium cannot in general be reciprocal.

Theorem 4. *Suppose* $G = \big(N,(S_i)_{i\in N},(u_i)_{i\in N}\big)$ *is a semi-symmetric link formation game with bilateral strategic complements such that: (i) derivative is strictly decreasing on the diagonal; (ii) parameters* $p_j > 0$ *and* $c_i > 0$ *of equation* (3) *are taken from compact intervals* P *and* C*, respectively; (iii) the function* V *in* (3) *is linear. Assume also that if all players* $i \in N$ *would have the same parameters* $p \in P, c \in C$*, then the corresponding symmetric game would have a symmetric interior equilibrium* s*. Then there exists an equilibrium with a complete network.*

Sketch of a proof. For each pair $p \in P$ and $c \in C$ of parameters there exists a symmetric interior equilibrium. By assumption (i) these equilibria can be naturally ordered. An equilibrium for G with a complete network can be formed from these symmetric equilibria. For details see Appendix. □

Remark 2. Note that Theorem 4 holds also if condition (i) is replaced by the condition that derivative is strictly increasing on the diagonal. In such a case an interior equilibrium is not stable in the usual best reply dynamics. The assumption of Theorem 4 that derivative is strictly decreasing (or strictly increasing) is critical as demonstrated in Example 1.

For games with bilateral strategic substitutes we have the following.

Theorem 5. *Suppose* $G = \big(N,(S_i)_{i\in N},(u_i)_{i\in N}\big)$ *is a semi-symmetric link formation game with bilateral strategic substitutes such that: (i) parameters* $p_j > 0$ *and* $c_i > 0$ *of equation* (3) *are taken from compact intervals* P *and* C*, respectively; (ii) the function* V *in* (3) *is linear. If for each* p_j *and* c_i*, and for each* $z \in (0,1/(n-1)]$ *there exists* $x \in (0,1/(n-1)]$ *such that* $p_j \partial U(x,z)/\partial x_1 - c_i = 0$*, then there exists an equilibrium with a complete network.*

Proof. See Appendix.

4.1 Efficiency and Stability of Equilibria

We have focused on equilibria such that the corresponding network is complete, or has complete components. It turns out in our framework completeness of equilibrium networks is in many cases closely related to stability and Pareto optimality of equilibria.

Given a game $G = \big(N,(S_i)_{i\in N},(u_i)_{i\in N}\big)$, a strategy profile s is Pareto optimal, if there is no other profile s' such that $u_i(s') \geq u_i(s)$ for all $i \in N$ and $u_j(s') > u_j(s)$ for some $j \in N$. A network corresponding to a strategy profile s is Pareto optimal, if s is a Pareto optimal strategy profile.

The following result gives conditions under which a Pareto optimal network must be complete. Intuitively, the condition that guarantees completeness of the equilibrium network is that the marginal utility from privacy is less than the marginal benefit from a sufficiently small reciprocal investment.

Proposition 1. *Suppose a link formation game $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ is such that for each $i \in N$ and $z^i \in (0,1]$ there exists $x^i \in (0,z^i)$ such that*

$$\frac{\partial U_{ij}(x^i,x^i)}{\partial x_1} > V_i'(z^i - x^i), \forall i,j \in N, i \neq j,$$

and that each U_{ij} is concave. If $s \in S$ is Pareto optimal and $s_{ii} > 0, \forall i \in N$, then $s_{ij}, s_{ji} > 0$, $\forall i,j \in N$.

Proof. Suppose to the contrary that $s \in S$ is Pareto optimal and $s_{ii} > 0, \forall i \in N$, but $s_{ij} = 0$ for some $i,j \in N$. Since $U_{ij}(0,s_{ji}) = 0$ and $U_{ji}(s_{ji},0) = 0$, Pareto optimality of s implies that $s_{ji} = 0$. By assumption, there exists $x^i < s_{ii}$ and $x^j < s_{jj}$ such that

$$\frac{\partial U_{ij}(x^i,x^i)}{\partial x_1} > V_i'(s_{ii} - x^i), \; \frac{\partial U_{ji}(x^j,x^j)}{\partial x_1} > V_j'(s_{jj} - x^j).$$

Since U_{ij} and V_i are concave functions, these inequalities hold for every $x \in (0, \min\{x^i, x^j\})$ as well. Given such an x, consider a strategy profile s' that is otherwise like the profile s, except that $s'_{ij} = s'_{ji} = x$, and $s'_{ii} = s_{ii} - x$, $s'_{jj} = s_{jj} - x$. Then $u_i(s') > u_i(s)$ and $u_j(s') > u_j(s)$ while $u_k(s') = u_k(s)$ for all $k \neq i,j$, and therefore s is not Pareto optimal, a contradiction. □

Remark 3. Proposition 1 holds for example when each U_{ij} is a strictly concave Cobb-Douglas function. The functions V_i can then be any concave, strictly increasing functions. Note that Proposition 1 holds also if functions U_{ij} have decreasing derivative on the diagonal, which is a weaker assumption than concavity.

An equilibrium s and the corresponding network are called s is *strongly pairwise stable*, if there is no strategy profile s' such that $u_i(s') > u_i(s)$ and $u_j(s') > u_j(s)$ for some $i,j \in N$, when $s_k = s'_k$ for all $k \in N \setminus \{i,j\}$ (Bloch and Dutta 2009). The following result states conditions such that the network corresponding to a strongly stable equilibrium must be complete.

Proposition 2. *Suppose a link formation game $G = \big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$ is such that for each $i \in N$ and $z^i \in (0,1]$ there exists $x^i \in (0,z^i)$ such that*

$$\frac{\partial U_{ij}(x^i,x^i)}{\partial x_1} > V_i'(z^i - x^i), \forall i,j \in N, i \neq j,$$

and that each U_{ij} is concave. If $s \in S$ is a strongly pairwise stable equilibrium and $s_{ii} > 0, \forall i \in N$, then $s_{ij}, s_{ji} > 0$, $\forall i,j \in N, i \neq j$.

Proof. The proof of Proposition 1 applies here. □

Remark 4. If the functions V_i satisfy $\lim_{z\to 0^+} V_i'(z) = +\infty$, then $s_{ii} > 0$ must hold at any equilibrium.

Remark 5. The main lesson of Propositions 1 and 2 is *not* that Pareto optimal networks are always complete, or that strongly stable equilibria have complete networks. Utilities from some links may be so low that these links are not formed either for efficiency or for equilibrium reasons. The lesson of these propositions is that network structures that have *complete components* often appear as efficient solutions or as equilibrium networks of a strongly pairwise stable equilibrium.

However, one can easily verify that if utility functions have the following Cobb-Douglas form, then *any* Pareto optimal or strongly pairwise stable equilibrium *must* have a complete network:

$$u_i(s) = \sum_{k \neq i} p_{ik} s_{ik}^{a_i} s_{ki}^{b_i} + c_i s_{ii}^{d_i},$$

where all parameters are strictly positive, $a_i + b_i < 1$, and $d_i < 1$.

In network literature efficiency is usually defined by using the utilitarian welfare function: those strategy profiles that maximize the sum of utilities are efficient. While such strategy profiles are Pareto optimal, not all Pareto optimal profiles satisfy this efficiency criterion.

If the functions U_{ij} are concave, then the utility functions u_i are concave on a simplex. In such a case each Pareto optimal strategy profile maximizes a *weighted* sum of players' utilities. The (positive) weights depend on the profile in question. If also the functions V_i satisfy $\lim_{z \to 0^+} V_i'(z) = +\infty$, then $s_{ii} > 0$ must hold at every Pareto optimal s, for all i.

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Appendix

Proof of Theorem 1. ($\Leftarrow$) A reciprocal equilibrium s is nontrivial if $s_{ij} = s_{ji} > 0$ for at least two players i, j. Let $N_1, \ldots, N_k$ be the complete components of the equilibrium network. If N_t has $m \geq 2$ members, then there exists $x = s_{ij}, i, j \in N_t$ such that

$$\frac{\partial U(x,x)}{\partial x_1} - V'(1-(m-1)x) \geq 0.$$

Since V is concave, the inequality (11) holds for this x.

($\Rightarrow$) Suppose that inequality (11) holds. Let m be the largest number, $m \leq n$, such that

$$\frac{\partial U(z,z)}{\partial x_1} - V'(1-(m-1)z) \geq 0$$

holds for some $z \in (0, 1/(m-1)]$. Clearly $m \geq 2$. Either there exists $z < 1/(m-1)$ such that this inequality is actually an equality, or else the inequality is satisfied by $z = 1/(m-1)$.

If $m = n$, then $s_{ij} = z$ for all $i, j \in N,\ j \neq i$, is a reciprocal equilibrium with a complete network. If $m < n$, then let k be the largest integer such that $km \leq n$. Choose k disjoint subsets N_t of N such that $|N_t| = m$ for all $t = 1, \ldots, k$. If the union of these subsets does not cover N, then let N_{k+1} be the residual subset.

Define $s \in S$ by setting $s_{ij} = z$ for all $i, j \in N_t,\ j \neq i,\ t = 1, \ldots, k$ (and also for $i, j \in N_{k+1}$ if this subset is nonempty) defines a nontrivial reciprocal equilibrium such that subsets N_t are complete components of the equilibrium network. □

Proof of Theorem 2. Since G is anonymous, $u_i = U_i + V_i$. By assumption, if all players have the same utility function u_i, there is a symmetric equilibrium s^i such that the resulting network is complete. Since u_i has constant derivative on the diagonal and V_i is strictly concave and increasing, the symmetric equilibrium s^i is unique. If every s^i is such that $s^i_{jk} = 1/(n-1)$, we are done. So we may assume that s^1 is the equilibrium in which $s^1_{ij} = x^1$ takes the smallest value, $i \neq j$, and $x^1 < 1/(n-1)$. Note that there may be another equilibrium s^k such that $s^k_{ij} = x^1$.

Construct a reciprocal equilibrium recursively as follows.

Step 1. Let N_1 be the subset of players for whom the following first order condition holds:

$$\frac{\partial U_i(x^1,x^1)}{\partial x} = V_i'\big(1-(n-1)x^1\big). \tag{A1}$$

By assumption, $|N_1| \geq 1$. If $N_1 = N$, the recursion ends. If $|N_1| < n$, then there exists at least one player for whom the left hand side of equation (A1) is greater than the right hand side.

Step 2. Let $x^2 \in (0,1)$ be the least number such that $x^1 < x^2$ and the following weak inequality is satisfied for at least one player:

$$\frac{\partial U_i(x^2,x^2)}{\partial x_1} \geq V_i'\big(1-n_1x^1-(n-n_1-1)x^2\big). \tag{A2}$$

Since the derivative of U_i is constant on the diagonal and V_i is strictly concave, such an x^2 exists uniquely. Let N_2 be the set of players for whom equation (A2) holds. If $|N_1|+|N_2|=n$, the recursion ends, because $N_1 \cap N_2 = \emptyset$. If $|N_1|+|N_2|<n$, continue the recursion to Step 3. Since there are n players, there is Step k, $k>2$, as follows.

Step k. Let $x^k \in (0,1)$ be the least number such that $x^{k-1} < x^k$ and the following weak inequality is satisfied for at least one player:

$$\frac{\partial U_i(x^k,x^k)}{\partial x} \geq V_i'\big(1-\sum_{t<k} n_t x^t - (n_k-1)x^k\big). \tag{A3}$$

By assumption and the previous Steps, such a number x^k exists uniquely. The subset N_k of players for whom equation (A3) holds, satisfies $|N_1|+\cdots+|N_k|=n$ and $\{N_1,\ldots,N_k\}$ is a partition of N.

Given player $i \in N$, let m be such that $i \in N_m$. Define $s_{ij}=x^t$ for all $j \neq i$ such that $j \in N_t$ and $t<m$. For $j \neq i$ such that $j \in N_t$ and $m \leq t$, let $s_{ij}=x^m$. Let $s_{ii} = 1-\sum_{t<m} n_t x^t - \big[\big(\sum_{m\leq t} n_t\big)-1\big]x^m$.

By construction s is a reciprocal equilibrium such that the resulting network is complete. □

Proof of Theorem 4. Denote the set of "types" of players by $T=P\times C$. Given any type $t=(p,c)\in T$, if all players had this type, then by assumption there exists a symmetric interior equilibrium s^t satisfying

$$\frac{p}{c}\frac{\partial U(x^t,x^t)}{\partial x_1}=1$$

where $s^t_{ij}=x^t$ and $s^t_{ii}=1-(n-1)x^t$ for all $i,j\in N, j\neq i$. Since derivative is strictly decreasing on the diagonal, these symmetric equilibria can be ordered so that $x^t > x^{t'}$ iff $p/c > p'/c'$, where $t=(p,c)$ and $t'=(p',c')$.

Let $\overline{p}$ and $\underline{p}$ be the greatest and least elements, respectively, of the interval P. Define analogously $\overline{c}$ and $\underline{c}$. So the symmetric equilibrium corresponding to the type $\overline{t}=(\overline{p},\underline{c})$ has the largest x^t, denoted by $\overline{x}$. The symmetric equilibrium corresponding to the type $\underline{t}=(\underline{p},\overline{c})$ has the least x^t, denoted by $\underline{x}$.

Suppose that there are k different types $t^1,\ldots,t^k \in T$ present in the player set N. Let N_m consists of all players whose type is $t^m, m=1,\ldots,k$.

Let us construct an equilibrium s with a complete network such that players in the same subset N_m treat each other reciprocally.

Step 1. Set $s_{ii}=y^{t^m}$, and $s_{ij}=x^{t^m}$, for all $i,j\in N_m$, for all $m=1,\ldots,k$. Note that the first order conditions of an interior equilibrium are satisfied by these choices. The values x^{t^m} and y^{t^m} are the same as in the symmetric equilibrium s^{t^m}.

Step 2. Take any players $i\in N_m$ and $j\in N_h$, $m\neq h$. Consider a two-person game with strategic complements between i and j. Let $t^m=(p,c)$ and $t^h=(p'c')$. Let b^t denote the best reply function of type $t=t^m,t^h$ against opponent's choices $x\in[\underline{x},\overline{x}]$. The best replies for these types (unique by strict concavity of $U(\cdot, x)$) satisfy

$$\frac{p'}{c}\frac{\partial U(b^{t^m}(x), x)}{\partial x_1} = 1 = \frac{p}{c'}\frac{\partial U(b^{t^h}(x), x)}{\partial x_1}.$$

If $p'/c = p/c'$, then best replies are the same. If $p'/c < p/c'$, then $b^{t^m}(x) < b^{t^h}(x)$. Since $\underline{p}/\overline{c} \leq p'/c < p/c' \leq \overline{p}/\underline{c}$, we have also $\underline{x} \leq b^t(\underline{x}) \leq b^t(\overline{x}) \leq \overline{x}$ for both types $t = t^m, t^h$. This holds since bilateral strategic complements implies $b^t(\underline{x}) \leq b^t(\overline{x})$ (increasing best reply function). Strictly decreasing derivative on the diagonal implies $\underline{x} \leq b^t(\underline{x})$ and $b^t(\overline{x}) \leq \overline{x}$, because $b^t(x^t) = x^t$. But then by Tarski's fixed point theorem the mapping $(x_m, x_h) \longrightarrow (b^{t^m}(x_h), b^{t^h}(x_m))$ on $[\underline{x}, \overline{x}] \times [\underline{x}, \overline{x}]$ has a fixed point (x^{mh}, x^{hm}).

Consider the game between all players in the set $N_m \cup N_h$. Then note that the choices y^{t^m}, x^{t^m}, x^{mh} for players in N_m and the choices y^{t^h}, x^{t^h}, x^{hm} for players in N_h form an equilibrium, since the resource constraints are satisfies by the definition of the symmetric equilibria $s^{t^1}, \ldots, s^{t^k}$, and the payoff of any player i is additively separable *w.r.t.* his opponents.

Since the types t^m and t^h were chosen arbitrarily, we have solved an equilibrium for the whole game. To see this, take any player i, and assume that $i \in N_m$. Then his choices satisfy the resource constraint:

$$y^{t^m} + |N_m - 1|x^{t^m} + \sum_{h \neq m} |N_h| x^{mh} = 1.$$

Since the first order conditions for maximum satisfied, we are done. □

Proof of Theorem 5. Let the "type set" be $T = P \times C$. Suppose that there are k different types $t^1, \ldots, t^k \in T$. Let N_m consist of all players whose type is $t^m, m = 1, \ldots, k$. We construct an equilibrium s such that players in the same subset N_{i_m} behave reciprocally.

Step 1. Suppose $i, j \in N_m$, so they both have the type $t^m = (p^m, c^m)$. Let $b^{t^m}(z)$ denote the unique best reply of either player to $z \in [0, 1/(n-1)]$. By assumption $b^{t^m}(1/(n-1)) \leq 1/(n-1)$. If equality holds, then $x^{t^m} = 1/(n-1)$ is a reciprocal equilibrium in the game with player set N_m.

Suppose $b^{t^m}(1/(n-1)) < 1/(n-1)$. Let $I^* = \{z \mid b^{t^m}(y) < y, \forall y \in [z, 1/(n-1)]\}$ and $x^* = \inf I^*$. Note that x^* exists since $1/(n-1) \in I^*$. We want to show that $b^{t^m}(x*) = x^*$.

By bilateral strategic substitutes, $b^{t^m}(1/(n-1) < b^{t^m}(z)$ for all $z \in I^*, z < 1/(n-1)$ and by assumption $b^{t^m}(1/(n-1)) > 0$. By the Theorem of the maximum, the best reply $b^{t^m}(z)$ is a continuous function on $[x^m - \varepsilon, 1/(n-1)]$, for any $\varepsilon > 0$ such that $x^* - \varepsilon > 0$. By continuity, $b^{t^m}(x^*) \leq x^*$. Again by continuity and the definition of I^*, this inequality cannot be strict, so $b^{t^m}(x^*) = x^*$. A reciprocal equilibrium in the game with player set N_m is obtained by setting $s_{ij} = x^* \equiv x^{t^m}$ for all $i, j \in N_m, i \neq j$.

Step 2. Suppose $i \in N_m$ and $j \in N_h$, $m \neq h$. Let $t^m = (p, c)$ and $t^h = (p', c')$. If $p'/c = p/c'$, then the choices given in *Step 1* apply. Given $x \in [0, 1/(n-1)]$, the best replies satisfy

$$\frac{p'}{c}\frac{\partial U(b^{t^m}(x),x)}{\partial x_1} = 1 = \frac{p}{c'}\frac{\partial U(b^{t^h}(x),x)}{\partial x_1}.$$

Now $b^{t^m}(x) < b^{t^h}(x)$ because $U(\cdot,x)$ is strictly concave function, and because $p'/c < p/c'$.

Consider the function $f(x) = b^{t^h}(b^{t^m}(x))$ on $[b^{t^m}(1/(n-1)), 1/(n-1)]$. This function is continuous, and $f(x) \leq 1/(n-1)$ for all x. At $x = b_i(1/(n-1))$, $f(x) \geq x$, since both best replies are decreasing functions. Hence there is a fixed point $x^{hm} = f(x^{hm})$. But then x^{hm} is the best reply of player j against $b^{t^m}(x^{hm}) = x^{mh}$, which in turn is the best reply of player i against x^{hm}.

Therefore $s_{ji} = x^{hm}, s_{ij} = x^{mh}$ forms an equilibrium when the player set is $N_m \cup N_h$.

Since the types t^m and t^h were chosen arbitrarily, we have solved an equilibrium for the whole game. To see this, take any player i, and assume that $i \in N_m$. Then his choices satisfy the resource constraint:

$$|N_m - 1|x^{t^m} + \sum_{h \neq m} |N_h| x^{mh} \leq 1.$$

Define $s_{ii} = y^{t^m}$ so that the resource constraint is satisfied as equality. Since the first order conditions for maximum satisfied, we are done. □