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The Iterative Nature of a Class of Economic Dynamics

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Abstract This work aims to demonstrate a rather specific "iterative nature" existing in a class of regular economic dynamics by revisiting two typical economic concepts as informative examples, viz., random utility and stochastic growth. We begin with a formal treatment of discrete dynamical system and its popular derivation, iterated function system, so that a solid foundation could be laid for our analysis of economic dynamics. Two economic systems afterwards are constructed to show how random utility function and stochastic growth in a classical economy could be essentially driven by some iterative elements. Besides, our analyses also implicitly show that a quite complex economic dynamics carrying substantial randomness could basically originate in some fairly simple dynamic principles.

Keywords Dynamical system, iterated function system, random utility function, stochastic growth, chaos
JEL classification C61, D99

1. Introduction

The present paper deals with economic dynamics in a very specific way with a quite general objective yet, that is, characterizing some critical and widely existent nature in a class of economic dynamics. A number of somewhat popular terms are usually adopted to describe economic dynamics, say "complex" and "chaotic" (cf., Goodwin 1990; Lorenz 1993; Tu 1994; Day 1994, 1999), which both convey that the basic mechanism of economic dynamics should be in essence highly hard to capture. In this work, we do not plan to argue this viewpoint, however do plan to see its negation, that some fairly simple economic principles could also generate complex or even chaotic properties.

The economic science used to study static models, and discuss their equilibria and comparative statics thereof. That being said, a great number of dynamic models have been developed, such as bifurcation phenomena in a delayed demand-supply system (cf., Leontief 1934; Kaldor 1934; Ezekiel 1938), chaotic properties in models of optimal economic growth (cf., Day 1983; Benhabib and Nishimura 1985; Boldrin and Montrucchio 1986), and nowadays many investigations on financial market dynamics. Evidently, the literature on economic dynamics, nonlinearity, and complexity is vast and also tends to be diverse, yet there is a lack of closely relevant ones to this article and hence we shall pass such potential references directly to our writing.

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The technical foundations are written in Section 2 and 3, and they are followed by two economic systems which in some sense are artificial. In Section 4, we study random utility function, and show different approaches of randomness aggregation in time preference. In Section 5, a classical economy driven by consumption and production is reconsidered. We show that a multiplicative shock in that economy could produce a stochastic growth which is determined equivalently by an iterated linear function system on one scaled real interval.

2. Discrete dynamical system

Throughout this article, we will use $\mathbb{R}^+$ to mean the nonnegative real numbers, and use $\mathbb{Z}^+$ and $\mathbb{Z}^-$ to mean the nonnegative and nonpositive integers. For any sets X and Y, $X \times Y$ denotes their Cartesian product. Let the state space and time domain be X and $\mathbb{Z}$, respectively. Suppose the state space X is a metric space with a metric $d : X \times X \to \mathbb{R}^+$.

Definition 1. A *discrete dynamical system* on X is a pair (X, f) with $x_{n+1} = f(x_n)$ for all $x_n, x_{n+1} \in X$ and all $n \in \mathbb{Z}$, where $f : X \to X$ is of class C^0.

The trajectory passing through a state $x \in X$ is

$$\gamma(x) = \{f^n(x) : n \in \mathbb{Z}\}, \tag{1}$$

and its positive and negative semi-trajectories are

$$\gamma^+(x) = \{f^n(x) : n \in \mathbb{Z}^+\}, \quad \gamma^-(x) = \{f^n(x) : n \in \mathbb{Z}^-\}.$$

Evidently, the positive semi-trajectory $\gamma^+(x)$ also represents the motion starting from the state x.

A state x is an *equilibrium state*, if $\gamma(x) = \{x\}$, or $f(x) = x$. A state y is an *ω-limit state* for an initial state x if $\lim_{n\uparrow+\infty} f^n(x) = y$, and the set of all such ω-limit states is called the *ω-limit set* of x, and denoted by $\omega(x)$. A set of states $S \subseteq X$ is *invariant* if $f(S) = S$. Note that any nonempty ω-limit set should be invariant, and thus we have $f\big(\omega(x)\big) = \omega(x)$ for all $\omega(x) \neq \emptyset$.

A set of states $A \subseteq X$ is an *attractor*, if there is a neighborhood $N(A,\varepsilon)$ such that $f\big(N(A,\varepsilon)\big) \subseteq N(A,\varepsilon)$ and

$$\omega\big(N(A,\varepsilon)\big) = \bigcap_{n\in\mathbb{Z}^+} f^n\big(N(A,\varepsilon)\big) = A,$$

but no proper subset of A has such properties (cf., Milnor 1985).

A state x (and also the motion $\gamma^+(x)$) is *periodic*, if there is a $k \in \mathbb{Z}^+$ such that $f^k(x) = x$, and the minimal $k \in \mathbb{Z}^+$ satisfying $f^k(x) = x$ is the *period* of $\gamma^+(x)$. If the period of $\gamma^+(x)$ is 1, then $f(x) = x$, and thus x is actually an equilibrium state. If the period of $\gamma^+(x)$ is $p < +\infty$, then

$$\gamma^+(x) = \big\{x, f(x), \ldots, f^p(x)\big\}.$$

A state x is called *finally periodic*, if there is an $m \in \mathbb{Z}^+$ such that $f^n(x)$ is a periodic state for all $n \geq m$, or equivalently stating, there is some $p \in \mathbb{Z}^+$ such that $f^{n+p}(x) = f^n(x)$ for all $n \geq m$. A state x is called *asymptotically periodic*, if there is a $y \in X$ such that

$$\lim_{n\uparrow+\infty} d\big(f^n(x), f^n(y)\big) = 0. \tag{2}$$

In case the state space $X \subseteq \mathbb{R}$ and it is compact, one would have the following theorem:

Theorem 1. (Li and Yorke 1975) *Suppose X is an interval in $\mathbb{R}$, and $f : X \to X$ is of class C^0. If there exists a motion of period 3 in (X, f), viz., there are three distinct states $x, y, z \in X$ such that $f(x) = y$, $f(y) = z$, and $f(z) = x$, then there is some motion of period n in (X, f) for all $n \in \mathbb{N}$.*

Proof. Let $<_S$ denote Šarkovskii's order on $\mathbb{N}$, then we have

$$3 <_S 5 <_S 7 <_S \cdots <_S 2^n <_S 2^{n-1} <_S \cdots <_S 2^2 <_S 2 <_S 1.$$

By Šarkovskii's (1964) theorem, if (X, f) has a motion of period m, then it must have some motion of period m' with $m <_S m'$. Since $3 <_S n$ for all $n \neq 3$, and there is a motion of period 3 in (X, f), the statement will thus directly follow. □

A generic dynamical system is chaotic if its dynamics sensitively depend on the initial state, and its states are transitive. For the moment, a discrete dynamical system (X, f) is called *chaotic* if it satisfies

(i) for all $x \in X$ and any $\varepsilon > 0$, there is a $\delta > 0$ such that $d\big(f^n(x), f^n(y)\big) > \varepsilon$ for all $y \in N(x, \delta)$ and some $n \in \mathbb{Z}^+$,

(ii) for all $S_1, S_2 \subseteq X$, there is an $n \in \mathbb{Z}^+$ such that $f^n(S_1) \cap S_2 \neq \emptyset$.

In particular, when $X \subseteq \mathbb{R}$ is compact, and f is of class C^0, an alternative definition of chaos can be proposed in the sense of Li and Yorke (1975).

Definition 2. A discrete dynamical system (X, f) is *nonperiodically chaotic*, if there is an uncountable set $S \subseteq X$ such that

(i) $\limsup_{n\uparrow+\infty} d\big(f^n(x), f^n(y)\big) > 0$ for all distinct $x, y \in S$,

(ii) $\liminf_{n\uparrow+\infty} d\big(f^n(x), f^n(y)\big) = 0$ for all distinct $x, y \in S$,

(iii) for all $z \in X$ periodic, $\limsup_{n\uparrow+\infty} d\big(f^n(x), f^n(z)\big) > 0$ for all $x \in S$.

It might be noticed that nonperiodic chaos is a slightly weaker concept than chaos itself. That's to say, if a discrete dynamical system on $X \subseteq \mathbb{R}$ is chaotic, then it must be nonperiodically chaotic as well; but if a discrete dynamical system is nonperiodically chaotic, it may not be chaotic.

The reason behind such an assertion is constructive. It (X, F) is nonperiodically chaotic, then there is at most one asymptotically periodic state in S. Now suppose a state $u \in X$ is not asymptotically periodic, then $\omega(u)$ should have infinitely many states. Let $V \subseteq \omega(u)$ be the (minimally invariant) kernel of $\omega(u)$, and suppose there is some $v \in X$ such that $V = \omega(v)$, which hence again contains infinitely many states. Let $U = X \setminus V$, then $f^n(V) \cap U = \emptyset$ for all $n \in \mathbb{Z}^+$, and therefore V and U are not transitive, which then implies (X, f) is not chaotic.

3. Iterated function system

Let's now consider a collection of contractive functions defined on the state space X with the metric d. Here, a function $f : X \to X$ is called *contractive*, if there is a $\lambda \in (0,1)$ such that $d\big(f(x), f(y)\big) \leq \lambda d(x,y)$ for all $x, y \in X$.

Let I_N denote an index set with N elements for $N \geq 2$ finite. Let

$$F = \{f_i : i \in I_N\},$$

where $f_i : X \to X$ is contractive and of class C^0 for all $i \in I_N$.

Definition 3. The pair (X, F) is called an *iterated function system*, if (X, f_i) is a discrete dynamical system for all $i \in I_N$.

Suppose X is compact, and let $\mathcal{Q}(X)$ denote the collection of all the nonempty compact subsets of X. Then $\mathcal{Q}(X)$ with the Hausdorff metric d_H is a compact metric space, where the Hausdorff metric d_H on $\mathcal{Q}(X)$ can be defined by the metric d on X, i.e., for all $U, V \in \mathcal{Q}(X)$

$$d_H(U,V) = \sup_{u \in U,\, v \in V} \big\{d(u,V), d(v,U)\big\},$$

in which $d(u,V) = \inf_{v \in V} d(u,v)$ and $d(v,U) = \inf_{u \in U} d(v,u)$.

Define a mapping $H : \mathcal{Q}(X) \to \mathcal{Q}(X)$, such that for all $B \in \mathcal{Q}(X)$,

$$H(B) = \bigcup_{i \in I_N} f_i(B). \tag{3}$$

Here, H is called the *Hutchinson operator* (Hutchinson 1981). Moreover, we define H^n by the recursion $H^n = H \circ H^{n-1}$ with $H^0 = \mathrm{id}_{\mathcal{Q}(X)}$, where $n \in \mathbb{Z}$ and $\mathrm{id}_{\mathcal{Q}(X)}$ denotes the identity mapping on $\mathcal{Q}(X)$.

Definition 4. $A \in \mathcal{Q}(X)$ is called an *attractor* of (X, F), if there is a neighborhood $N(A, \varepsilon) \in \mathcal{Q}(X)$ such that

$$H\big(N(A,\varepsilon)\big) \subseteq N(A,\varepsilon), \qquad \bigcap_{n \in \mathbb{Z}^+} H^n\big(N(A,\varepsilon)\big) = A,$$

and no proper subset of A in $\mathcal{Q}(X)$ has such properties.

Theorem 2. *(X, F) has a unique attractor A with $H(A) = A$.*

Proof. For all $f_i \in F$, there is a $\lambda_i \in (0,1)$ such that for all $x, y \in X$,

$$d\big(f_i(x), f_i(y)\big) \leq \lambda_i d(x,y).$$

Let $\lambda = \max_{i \in I_N} \lambda_i$, then $\lambda \in (0,1)$ as well. Note that for all $U, V \in \mathcal{Q}(X)$ we have

$$d_H\big(H(U), H(V)\big) \leq \sup_{i \in I_N} d_H\big(f_i(U), f_i(V)\big) \leq \sup_{i \in I_N} \lambda_i d_H(U,V) \leq \lambda d_H(U,V),$$

thus by the Banach fixed point theorem, there is a unique $A \in \mathcal{Q}(X)$ such that $H(A) = A$, and $\lim_{n \uparrow +\infty} H^n(B) = A$ for all $B \in \mathcal{Q}(X)$. And clearly, there exists a neighborhood $N(A, \varepsilon) \in \mathcal{Q}(X)$ serving as a basin of A.

We then show that any $B \neq A$ in $\mathcal{Q}(X)$ can not be an attractor of (X,F), which would imply A is the unique attractor of (X,F), and thus completes our proof. First of all, any $B \supset A$ can not be an attractor of (X,F), as for all $\varepsilon > 0$

$$\bigcap_{n \in \mathbb{Z}^+} H^n\big(N(B, \varepsilon)\big) \subseteq A \subset B.$$

Next, any $B \subset A$ also can not be an attractor of (X,F), otherwise we would have

$$\lim_{n \uparrow +\infty} H^n\big(N(B, \varepsilon)\big) = B \subset A,$$

a contradiction. □

Now consider the space I_N^ω, and for all $\mu \in I_N^\omega$ we write

$$\mu = (\mu_n,\ n \in \mathbb{N}) = (\mu_1, \mu_2, \ldots, \mu_\omega),$$

where $\mu_n \in I_N$ for all $n \in \mathbb{N}$. The Baire metric between all $\mu, \upsilon \in I_N^\omega$ is

$$d_B(\mu, \upsilon) = 2^{-m},$$

where $m = \min\{n \in \mathbb{N} : \mu_n \neq \upsilon_n\}$. Clearly, (I_N^ω, d_B) is a compact metric space. Let's define a mapping $C : I_N^\omega \times \mathcal{Q}(X) \to \mathcal{Q}(X)$, such that for all $\mu \in I_N^\omega$ and $S \in \mathcal{Q}(X)$,

$$C(\mu, S) = \bigcap_{n \in \mathbb{N}} f_{\mu_\omega} \circ \cdots \circ f_{\mu_{n+1}} \circ f_{\mu_n}(S). \tag{4}$$

Note in addition that the motion of any state $x \in S$ can be expressed as

$$\gamma^+(x) = \{f_{\mu_n} \circ \cdots \circ f_{\mu_2} \circ f_{\mu_1}(x) : n \in \mathbb{N}\}.$$

Suppose $B(A) = N(A, \varepsilon)$ for some $\varepsilon > 0$ is a basin of the attractor A, then for all $S \subseteq B(A)$ and $\mu \in I_N^\omega$, we have $C(\mu, S) \subseteq A$, and hence one can write

$$C\big(I_N^\omega, B(A)\big) = A. \tag{5}$$

It therefore suggests that the attractor A of (X,F) could be practically attained by all the ω-permutations of the transition rules in F.

Suppose there is some probability measure on I_N^ω, and in particular, we shall assume it is stationary, so that it can be fully characterized by a discrete probability measure on I_N. Let $\pi : I_N \to [0,1]$ denote such a probability measure, which satisfies $\sum_{i \in I_N} \pi(i) = 1$. As a consequence, at any time a function f_i stands out in F with a probability $\pi(i)$ for all $i \in I_N$.

Definition 5. The triplet (X,F,π) is called an *iterated random function system*.

Let σ_n be a random variable, such that $\mathrm{Prob}(\sigma_n = i) = \pi(i)$ for all $i \in I_N$. The transition function at a time $n \in \mathbb{Z}$ can thus be denoted by a randomly indexed function f_{σ_n}. Let a random variable Z_n denote the stochastic state in the system (X,F,π) at the time $n \in \mathbb{Z}$, then we have

$$Z_{n+1} = f_{\sigma_{n+1}}(Z_n). \tag{6}$$

Suppose the initial time is 0, and the initial state is $x \in X$, then the random motion can be written as

$$\Gamma^+(x) = \{Z_n : n \in \mathbb{Z}^+\},$$

in which $Z_0 = x$, $Z_1 = f_{\sigma_1}(x)$, and $Z_n = f_{\sigma_n}(Z_{n-1})$ for all $n \geq 2$.

Note that the stochastic process $(Z_n,\ n \in \mathbb{N})$ is in effect a Markov chain, and it is equivalent to the iterated random function system (X,F,π) (cf., Diaconis and Freedman 1999). Suppose $Z_n = z$, then $\mathrm{Prob}(Z_{n+1} \in S)$ for some $S \subseteq X$ takes the following value

$$P(z,S) = \sum_{i \in I_N} \pi(i) \mathbf{1}_S\big(f_i(z)\big), \tag{7}$$

where the characteristic function $\mathbf{1}_S$ is defined as

$$\mathbf{1}_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

When S is a Borel subset of X, there should be an invariant probability measure ρ, such that

$$\rho(S) = \int_X P(z,S)\, d\rho(z) = \sum_{i \in I_N} \pi(i) \rho\big(f_i^{-1}(S)\big). \tag{8}$$

Here, ρ is called a *π-balanced measure* for (X,F,π), as was proposed by Barnsley and Demko (1985).

Let $R(\rho)$ denote the support of ρ, then $R(\rho) = \{x \in X : \rho(x) \neq 0\}$ and hence

$$R(\rho) = \bigcup_{i \in I_N} f_i\big(R(\rho)\big) = H\big(R(\rho)\big). \tag{9}$$

By Theorem 2, it directly appears that $R(\rho) = A$, and therefore the support of a π-balanced measure for (X,F,π) is exactly the unique attractor A of (X,F) for all π. As a result, we can see that the attractor A of (X,F) can also be attained by a random ω-permutation of transition rules in F.

Example 1. Let's assume $X = [0,1]$, $I_2 = \{a,b\}$, $\pi(a) = \pi(b) = 0.5$, and $F = \{f_a, f_b\}$ with

$$f_a : x \mapsto x/3, \quad f_b : x \mapsto x/3 + 2/3.$$

The iterated random function system (X,F,π) is clearly equivalent to the following autoregressive process,

$$Z_{n+1} = Z_n/3 + \varepsilon_{n+1} \quad (n \in \mathbb{Z}^+), \tag{10}$$

where Z_0 is deterministic, and for all $n \in \mathbb{N}$

$$\text{Prob}(\varepsilon_n = 0) = \text{Prob}(\varepsilon_n = 2/3) = 0.5.$$

The iterated function system (X,F) has a unique attractor A as a Cantor ternary set, that is,

$$A = \Big\{ \sum_{n\in\mathbb{N}} x_n/3^n : (x_n,\ n \in \mathbb{N}) \in \{0,2\}^\omega \Big\}. \tag{11}$$

Let

$$B_k = \Big\{ \sum_{n\geq k} x_n/3^n : (x_n,\ n \geq k) \in \{0,2\}^\omega \Big\} \quad (k \in \mathbb{N}),$$

then $B_1 = A$ and $B_2 = A/3$, and thus

$$f_a(A) = B_2, \quad f_b(A) = B_2 + 2/3,$$

which yield $f_a(A) \cup f_b(A) = B_1$. Recall that $H = f_a \cup f_b$ is the Hutchinson operator for (X,F), one can thus write $H(A) = A$.

In practice, the attractor A can be realized by a random motion with any initial state $x \in [0,1]$ in (X,F,π). There are two cases to consider.

If $x \in A$, then $\Gamma^+(x) = A$ almost surely. If $x \notin A$, then there should be a finite sequence $(x_1, x_2, \ldots, x_k) \in \{0,2\}^k$, such that

$$x = \sum_{n=1}^{k} x_n/3^n + r_k(x),$$

where $r_k(x) \leq 1/3^k$. Clearly, $r_k(x)$ will tend to be 0 when k goes to infinity. Now we have $Z_1 = x/3 + \varepsilon_1 = Y_1 + r_k(x)/3$, where $Y_1 = \sum_{n=1}^{k} x_n/3^{n+1} + \varepsilon_1 \in A$, and in general, $Z_m = Y_m + r_k(x)/3^m$, where $Y_m \in A$ and $r_k(x)/3^m \leq 1/3^{m+k}$. Evidently, there should be an ℓ such that $Z_\ell \in A$, which suggests $\Gamma^+(Z_\ell) = A$ almost surely.

4. Random utility

Consider a generic agent w in a large group W, and suppose w has a preference relation by nature. Let X denote a decision state space for the group W, and let $\succsim$ be a weak order on X such that

(i) either $x \succsim y$ or $y \succsim x$ for all $x, y \in X$,

(ii) $x \gtrsim y$ and $y \gtrsim z$ implies $x \gtrsim z$ for all $x,y,z \in X$.

So $\gtrsim$ can serve as a *rational* preference relation for w. In particular, we shall assume that there is a utility function $u : X \to \mathbb{R}$ such that for all $x,y \in X$,

$$x \gtrsim y \Longleftrightarrow u(x) \geq u(y).$$

Let $\mathcal{P}(X)$ be the power set of X. A mapping $C : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a *choice function* if $\emptyset \neq C(Y) \subseteq Y$ for all nonempty $Y \in \mathcal{P}(X)$.

If $y \in C(Y)$ for some $y \in Y$, and meanwhile, $u(y) \geq u(x)$ for all $x \in Y$, we say the choice made by w matches to her preference relation. It should be noted that there are two implicit assumptions under this statement, i.e., the choices made by w can be perfectly observed, and w can perfectly identify and also intentionally apply her preference relation. However, it seems that empirical verifications would be unable to simultaneously support these two intertwined assumptions. The reason is that observations about the choices made by w are reasonable only if w does make her choices complying with her preference relation, and on the other hand, the true preference relation of w can be thought of as identifiable only if observations about her choices are perfect.

To overcome such difficulties in empirical verifications of consistency of choice and preference, we have to set one assumption ad hoc true, so that we could verify the other one. To begin with, if the preference relation of w is supposed to be perfectly identifiable and intentionally applied by w herself, then it will become possible to infer it from observations about her choices with some confidence level. This approach appeared in a study on stochastic utility model estimation by Manski (1975).

Let $v : X \to \mathbb{R}$ denote a utility function consistent with observations about the choices made by w. And we shall say $v(x)$ is the observed utility if a choice $x \in X$ has been observed. It thus appears to us that

$$u(x) = v(x) + \varepsilon(x), \tag{12}$$

where $\varepsilon(x)$ denotes a “noise” function that might be independently distributed for all $x \in X$. In particular, the choice x can be assumed to be characterized by n independently observed attributes, $J(x) \in \mathbb{R}^n$, thus $v(x)$ admits a linearly parametric model $v(x) = \beta' J(x)$ for $\beta \in \mathbb{R}^n$. In consequence, we have

$$u(x) = \beta' J(x) + \varepsilon(x), \tag{12$'$}$$

in which the estimation $\hat{\beta}$ is determined by the observed data $J(x)$ for $x \in S$, where $S \subseteq X$ is a certain sample.

On the other hand, if the choices made by w are supposed to be perfectly observed, then we could discuss her identification of the true preference relation. In practice, the true preference relation might be only partially identified by w, but it should not be totally vague to her, even if she had an extremely limited cognitive ability. Suppose w has a collection of observable utility functions, which can represent her identified preference relations in different situations, and all these utility functions have an identical kernel as her invariant knowledge of her true preference relation.

Let I_N be a finite index set with $|I_N| = N \geq 2$, and let $v_i : X \to \mathbb{R}$ be a utility function of w for all $i \in I_N$. Suppose $u : X \to \mathbb{R}$ is the kernel utility function of all v_i for $i \in I_N$. Let $X_u = u(X)$, then $X_u \subseteq \mathbb{R}$. And for all $i \in I_N$, there is a contractive function $f_i : X_u \to X_u$ such that $v_i = fi \circ u$, or $v_i(x) = f_i\big(u(x)\big)$ for all $x \in X$. Clearly, $\{v_i : i \in I_N\}$ on the domain X is now equivalent to $F = \{f_i : i \in I_N\}$ on the domain X_u.

Suppose w makes her choices along the time domain $\mathbb{Z}^+$ in such a way that at each time $t \in \mathbb{N}$, she picks a function $f_i \in F$ to form her utility function

$$u_t = f_i \circ u_{t-1}, \tag{13}$$

where u_{t-1} is her utility function at the time $t-1$. More concretely, at the initial time 0, the utility function of w is set as her kernel utility function, i.e., $u_0(x) = u(x)$, and at the time 1, her utility function is $u_1(x) = f_i\big(u(x)\big)$ for some $i \in I_N$. In general, at any time $t \in \mathbb{N}$, her utility function is $u_t(x) = f_i\big(u_{t-1}(x)\big)$ for some $i \in I_N$. Here, the sequence of utility functions $\big(u_t(x),\, t \in \mathbb{Z}^+\big)$ can be considered as a general extension of a normal discounted utility function series, and in terms of time preference, we actually generalize $(\succsim,\, t \in \mathbb{Z}^+)$ to $(\succsim_t,\, t \in \mathbb{Z}^+)$, where $\succsim_t$ varies across time.

For the moment, we should notice that $(u_t(x),\, t \in \mathbb{Z}^+)$ is completely determined by the iterated function system (X_u, F). By Theorem 2, one can directly see that (X_u, F) has a unique attractor, say $A \subseteq X_u$, such that $A = \bigcup_{i \in I_N} f_i(A)$. It thus suggests that some kernel utilities in A could be reached by w in the long term.

Let π denote a probability measure on I_N, then an iterated random function system (X_u, F, π) will emerge. By (6), we obtain

$$U_{t+1}(x) = f_{\sigma_{t+1}}\big(U_t(x)\big) \quad (t \in \mathbb{Z}^+), \tag{14}$$

where $U_0(x) = u(x)$, and $U_t(x)$ denotes the random utility function at the time t.

If $f_i(x) = \rho_i x$ for all $i \in I_N$, where $\rho_i \in (0,1)$ and $\rho_i \neq \rho_j$ for all distinct $i, j \in I_N$, then (14) will be

$$U_{t+1}(x) = \xi_{t+1} U_t(x) \quad (t \in \mathbb{Z}^+), \tag{14$'$}$$

where $\text{Prob}(\xi_t = \rho_i) = \pi(i)$ for all $i \in I_n$ and all $t \in \mathbb{N}$. Thus at any time $t \in \mathbb{N}$, the random utility function of w is

$$U_t(x) = \Big(\prod_{n=1}^{t} \xi_n\Big) u(x) = \exp\Big(\sum_{n=1}^{t} \log \xi_n\Big) u(x) = \exp\Big(-\sum_{n=1}^{t} \log(1/\xi_n)\Big) u(x).$$

Let $\delta_t = \sum_{n=1}^{t} \log(1/\xi_n)$, then the random utility function of w at $t \in \mathbb{N}$ can be written as

$$U_t(x) = \mathrm{e}^{-\delta_t} u(x). \tag{15}$$

When t goes to infinity, δ_t will approach infinity, and thus $U_t(x)$ will approach zero almost surely for all choice x.

If $f_i(x) = \rho x + r_i$ for all $i \in I_N$, where $\rho \in (0,1)$, $r_i > 0$, and $r_i \neq r_j$ for all distinct $i, j \in I_N$, then (14) will be

$$U_{t+1}(x) = \rho U_t(x) + \theta_{t+1} \quad (t \in \mathbb{Z}^+), \tag{14''}$$

in which once again $\text{Prob}(\theta_t = r_i) = \pi(i)$. At any time $t \in \mathbb{N}$, the random utility function of w then becomes

$$U_t(x) = \rho^t u(x) + \sum_{n=1}^{t} \rho^{t-n} \theta_n. \tag{16}$$

Note that $\rho^t u(x)$ will vanish when t goes to infinity, but the remaining part will not converge almost surely, as a new piece of randomness θ_t will emerge at each time t.

5. Stochastic growth

Consider an economy with a production function $Y = F(K,L)$, where Y, K, L denote the total production, the capital input, and the labor supply in the economy, respectively. Let $y = Y/L$ and $k = K/L$, and suppose $F(K,L)$ is a homogeneous function of degree 1, then $Y/L = F(K/L, 1)$. Define $f(k) = F(K/L, 1)$, thus the production technology of a generic agent w in that economy can be represented by

$$y = f(k) \quad (k \in \mathbb{R}^+). \tag{17}$$

As typically assumed, $f(k)$ should satisfy that for all $k \in \mathbb{R}^+$

$$f'(k) > 0, \quad f''(k) < 0,$$

and the following Inada conditions, which are usually named after K. Inada, but also partly attributed to H. Uzawa (Uzawa 1961),

$$\lim_{k \downarrow 0} f'(k) = +\infty, \quad \lim_{k \uparrow +\infty} f'(k) = 0.$$

Let's now introduce a stochastic factor ξ into the economy, so that the production technology of w can be expressed as

$$y = f(k, \xi) \quad (k \in \mathbb{R}^+). \tag{18}$$

In case k and ξ are separable, we could consider two fundamental cases, i.e., ξ is an additive shock to $f(k)$, or ξ is a multiplicative shock to $f(k)$. Similar to the studies by Mitra et al. (2004), and Mitra and Privileggi (2009), we shall also focus on the latter case, and rewrite the technology (18) as

$$y = \xi f(k) \quad (k \in \mathbb{R}^+), \tag{18'}$$

where $\xi > 0$ is a random variable. In practice, we can assume that the support of ξ is $\{\lambda_i : i \in I_N\}$, where I_N is a finite index set with $|I_N| = N \geq 2$, and there is a probability measure π on I_N, such that $\mathrm{Prob}(\xi = \lambda_i) = \pi(i)$ for all $i \in I_N$.

In addition, the consumption and investment which are both necessary parts of a sustainable economy, are denoted by C and E, thus we should have $Y = C + E$. Let $c = C/L$ and $e = E/L$, then the income identity for w is $y = c + e$. Suppose the economy functions on the time domain $\mathbb{Z}^+$, so that the economic variables all become discretely time-dependent, that is, $y_t, k_t, c_t, e_t, \xi_t$ for $t \in \mathbb{Z}^+$, then the economy can be represented by the following system:

$$\begin{cases} y_t = \xi_t f(k_t) \\ y_t = c_t + e_t \\ k_{t+1} = e_t \end{cases}$$

in which $k_0 \neq 0$ is the initial capital input, and $\xi_t, \xi_{t'}$ are independent for all distinct $t, t' \in \mathbb{Z}^+$.

Suppose w has a stationary utility function in her consumption c which is written as $u(c)$, such that $u'(c) > 0$ and $u''(c) < 0$ for all $c \in \mathbb{R}^+$, and $\lim_{c \downarrow 0} u'(c) = +\infty$, then it clearly appears that $c_t > 0$ at any time $t \in \mathbb{Z}^+$. Assume the time preference of w can be characterized by a regular discounting $\rho \in (0,1)$, then her additive utilities from a deterministic consumption flow $(c_0, c_1, \dots, c_t)$ for $t \in \mathbb{Z}^+$, can be expressed as $\sum_{n=0}^{t} \rho^n u(c_n)$.

The steady growth path of the economy is thus determined by the equilibrium of the decision-making process for w. In other words, w maximizes

$$\mathrm{E}_0 \sum_{t \in \mathbb{Z}^+} \rho^t u(c_t),$$

subject to $c_t = \xi_t f(k_t) - k_{t+1}$ for all $t \in \mathbb{Z}^+$ with $k_0 > 0$ initially given. Here, we apply E_t to denote the expectation operator at a time $t \in \mathbb{Z}^+$.

Recall that an optimal consumption flow $(c_t,\ t \in \mathbb{Z}^+)$ should satisfy the following Euler equation,

$$u'(c_t) = \rho\, \mathrm{E}_t \big(\xi_{t+1} f'(k_{t+1}) u'(c_{t+1})\big). \tag{19}$$

Since $k_{t+1} = y_t - c_t$, (19) is equivalent to

$$u'(c_t) = \rho f'(y_t - c_t)\, \mathrm{E}_t \big(\xi_{t+1} u'(c_{t+1})\big). \tag{19$'$}$$

There should be a real function φ such that $c_t = \varphi(y_t)$ for all c_t in the optimal consumption flow, which yields $k_{t+1} = y_t - \varphi(y_t)$, and thus

$$y_{t+1} = \xi_{t+1} f(k_{t+1}) = \xi_{t+1} f\big(y_t - \varphi(y_t)\big).$$

Let $\psi(y) = f\big(y - \varphi(y)\big)$, then we have the following stochastic growth process:

$$y_{t+1} = \xi_{t+1} \psi(y_t) \quad (t \in \mathbb{Z}^+). \tag{20}$$

Let $X_Y \subseteq \mathbb{R}^+$ be an invariant support set for y_t driven by the above process (20), so that $y_t \in X_Y$ at any $t \in \mathbb{Z}^+$. Define $g_i(y) = \lambda_i \psi(y)$ for all $y \in X_Y$. Let $G = \{g_i : i \in I_N\}$, then the stochastic growth process $(y_t,\ t \in \mathbb{Z}^+)$ as is determined by (20) should be equivalent to the iterated random function system (X_Y, G, π).

Corresponding to the optimal consumption flow, the following optimal capital flow would directly come out,

$$k_{t+1} = y_t - \varphi(y_t) = \xi_t f(k_t) - \varphi\big(\xi_t f(k_t)\big), \tag{21}$$

which can also be supposed to admit an invariant support set $X_K \subseteq \mathbb{R}^+$. Define

$$m_i(k) = \lambda_i f(k) - \varphi\big(\lambda_i f(k)\big),$$

and let $M = \{m_i : i \in I_N\}$, then we have another iterated random function system (X_K, M, π), which in a sense is conjugate to the former (X_Y, G, π).

Example 2. Take $I_N = \{a, b\}$, $f(k) = \sqrt[3]{k}$, and $u(c) = \log c$. Let's suppose $(\xi_t,\, t \in \mathbb{Z}^+)$ is a Bernoulli process with

$$\text{Prob}(\xi_t = \lambda_a) = q, \quad \text{Prob}(\xi_t = \lambda_b) = 1 - q,$$

where $q \in (0, 1)$, and

$$1/\lambda_a^2 < \lambda_b < 1 < \lambda_a < 1/\lambda_b.$$

It thus suggests that the shock is either positive or negative, while the negative shock would not make the economy vanish as $\lambda_b \lambda_a^2 > 1$, and the positive shock would not make it too expansive as $\lambda_a \lambda_b < 1$.

In the optimal consumption flow $(c_t,\, t \in \mathbb{Z}^+)$, we might see that $c_t = (1 - \rho/3) y_t$, which yields $\varphi(y_t) = (1 - \rho/3) y_t$, and thus the optimal capital flow is determined by the formula

$$k_{t+1} = \rho y_t / 3 = \rho \xi_t \sqrt[3]{k_t} / 3.$$

Let $\kappa_t = \log k_t$, then we have

$$\kappa_{t+1} = \kappa_t / 3 + \log \xi_t + \log(\rho/3),$$

which should have an invariant support interval $[\alpha, \beta] \subset \mathbb{R}$.

We now have the following two affine functions:

$$\ell_a(\kappa) = \kappa/3 + \big(\log \lambda_a + \log(\rho/3)\big), \quad \ell_b(\kappa) = \kappa/3 + \big(\log \lambda_b + \log(\rho/3)\big).$$

Let $\Lambda = \{\ell_a, \ell_b\}$, then $([\alpha, \beta], \Lambda)$ is an iterated function system. Notice that

$$\beta/3 + \big(\log \lambda_a + \log(\rho/3)\big) = \beta, \quad \alpha/3 + \big(\log \lambda_b + \log(\rho/3)\big) = \alpha,$$

so $\log \lambda_a + \log(\rho/3) = 2\beta/3$ and $\log \lambda_b + \log(\rho/3) = 2\alpha/3$, and thus $\ell_a(\kappa)$ and $\ell_b(\kappa)$ can be also written as

$$\ell_a(\kappa) = \kappa/3 + 2\beta/3, \quad \ell_b(\kappa) = \kappa/3 + 2\alpha/3,$$

where $\beta > \alpha$ because $\lambda_a > \lambda_b$. Let $z = (\kappa - \alpha)/(\beta - \alpha)$, then Λ on $[\alpha, \beta]$ can be transformed into a pair of functions defined on $[0,1]$, i.e.,

$$Z = \{z/3, z/3 + 2/3\}.$$

It therefore appears that $([\alpha,\beta],\Lambda)$ is equivalent to the iterated function system $([0,1],Z)$. By Example 1, we know that the unique attractor of $([0,1],Z)$ is the Cantor ternary set, and thus the attractor of $([\alpha,\beta],\Lambda)$ should be also a Cantor set, which then conveys that the dynamics of the optimal stochastic growth in the economy should be essentially chaotic.

6. Concluding remarks

In this article, we have demonstrated by example how a plain mechanism could generate a complex economic system with increasing disorder through the iteration process. In reality, economic systems usually themselves show very complicated dynamics which could be observed and recorded. For example, the quote dynamics in a security market are erratic and occasionally trapped in catastrophes. People are inclined to understand such "irregular" phenomena from the statistical viewpoint, that is, the dynamics should be replicated by a certain skeleton with some additional randomness or perturbation. With a flavor of this work, one might perceive a quite different approach to interpret the irregularity, that is, the dynamics might be driven by some deterministic rules, and the irregularity emerges as a form of chaos.

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