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Robust Turnpikes Deduced by the Minimum-Time Needed toward Economic Maturity

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Abstract In the paper, a one-sector neoclassical model with stochastic growth has been constructed. The major goal of the study is to characterize relevant mathematical properties of efficient development paths for underdeveloped economies. Since economic maturity is a reasonable objective, we mainly focus on the long-run features of economic development. Indeed, the notion of economic maturity is well-defined in the model, and also a thorough characterization of the minimum time needed toward economic maturity is offered with intuitive interpretations discussed. Moreover, it is confirmed that the capital-labor ratio corresponding to the state of economic maturity provides us with a robust turnpike of the optimal path of capital accumulation.

Keywords Stochastic growth, economic maturity, minimum-time objective, asymptotic turnpike theorem, neighborhood turnpike theorem, robustness
JEL classification C60, E13, E22

1. Introduction

When concerning the issue of economic development for underdeveloped economies, the principle of maximum speed is widely employed. In reality, the Germany and Japan after World War II and China after 1978s (see Song et al. 2011) are typical examples.

Alternatively, provided the existence of maximum sustainable terminal path consumption per capita (or von Neumann path consumption per capita), which would be regarded as the state of economic maturity in a certain sense, the major goal of people and government is to choose appropriate or optimal savings strategy and fiscal policies, respectively, such that the state of economic maturity can be reached as soon as possible. Indeed, the underlying motivation of the present exploration, which is in line with Kurz (1965), is to derive conditions under which the specified economy can reach the maximum terminal path in a minimum time. In particular, we analyze the economy before reaching economic maturity, and hence we focus on underdeveloped economies and leave those economies having reached economic maturity to future research.

Although we focus on a one-sector neoclassical aggregate growth model (see Solow 1956; Cass 1965; Dai 2013, 2014a, 2014c), the present study extends Kurz's analyses in the following ways. First, we consider an economy lying in a persistently non-stationary environment. Second, nature (or social planner) is incorporated into the

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macroeconomic model, and the endogenous savings rate and the minimum time form a sub-game perfect Nash equilibrium (SPNE) of the stochastic dynamic game between the nature and the representative agent. Third, the minimum time needed to reach economic maturity is completely characterized by the maximum sustainable level of terminal path capital-labor ratio (i.e., the state corresponding to economic maturity), and also the terminal path of capital-labor ratio provides us with a robust turnpike (i.e., the equilibrium path of capital accumulation will robustly converge to this terminal path in an asymptotic sense or will spend almost all time staying in a neighborhood of the terminal path). In addition, rather than letting the terminal capital-labor ratio be exogenously given or prescribed as in Kurz (1965), Samuelson (1965) and Cass (1966), the maximum sustainable level of terminal path consumption per capita (or capital-labor ratio) is endogenously determined in the present model.

The rest of the paper is organized as follows. In Section 2, the basic model is constructed, and some necessary assumptions and definitions, especially the definitions of economic maturity and the minimum time needed to economic maturity, are introduced. Section 3 is the major part of the paper, where both Asymptotic Turnpike Theorem and Neighborhood Turnpike Theorem are established. Section 4 proves robustness of the turnpike theorems established in Section 3, i.e., we assert the existence of a robust turnpike deduced by the minimum-time needed to economic maturity. There is a brief concluding section, where we have discussed possible extensions of the basic framework. All proofs, unless otherwise noted in the text, appear in the Appendix.

2. The environment

Here, and throughout the paper, we consider a one-sector neoclassical model with stochastic growth. As usual, we employ the following neoclassical production function

$$Y(t) = F\left(K(t), L(t)\right), \tag{1}$$

which is a strictly concave function and exhibits constant returns to scale (CRS) with $K(t)$ denoting the aggregate capital stock and $L(t)$ representing the labor force (or population size in some cases). Thus, we have the following law of motion of capital accumulation

$$\frac{dK(t)}{dt} = F\left(K(t), L(t)\right) - \delta K(t) - C(t), \tag{2}$$

where δ, an exogenously given constant, denotes the depreciation rate and $C(t)$ stands for aggregate consumption in period t.

Suppose that $(B(t),\ 0 \le t \le T)$ stands for a standard Brownian motion defined on the following filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ with $\mathbb{F} \equiv \{\mathcal{F}_t\}_{0\le t\le T}$ the $\mathbb{P}$-augmented filtration generated by $(B(t),\ 0 \le t \le T)$ with $\mathcal{F} \equiv \mathcal{F}_T$ for $\forall T > 0$, i.e., the underlying stochastic basis satisfies the well-known usual conditions. Then, based upon the given probability space and following Merton (1975) and Dai (2014a), we define the following law of motion for labor force

$$dL(t) = nL(t)dt + \sigma L(t)dB(t) \tag{3}$$

subject to $B(0) = 0$ almost surely (hereafter a.s.)-$\mathbb{P}$ and $\sigma \in \Re_0 \equiv \Re - \{0\}$, a constant. Thus, combining (2) with (3) and applying Itô's rule lead us to

$$dk(t) = \left[s(k(t)) f(k(t)) - \left(\delta + n - \sigma^2\right) k(t)\right] dt - \sigma k(t) dB(t) \tag{4}$$

with $k(0) \equiv k_0 > 0,\ k(t) \equiv K(t)/L(t),\ f(k(t)) \equiv F(K(t), L(t))/L(t) = F\left(\frac{K(t)}{L(t)}, 1\right)$, $s(k(t)) \equiv 1 - \frac{c(t)}{f(k(t))}$ and $c(t) \equiv C(t)/L(t)$ denoting the initial capital-labor ratio, capital-labor ratio, per capita output, savings per unit output and per capita consumption, respectively, at time t. Specifically, for the SDE of capital-labor ratio given by (4), Chang and Malliaris (1987) proved the following theorem.

Theorem 1. *If the production function f is strictly concave, continuously differentiable on $[0, \infty)$, $f(0) = 0$, and $\lim_{k(t)\to\infty} f'(k(t)) \equiv \lim_{k(t)\to\infty} \frac{df(k(t))}{dk(t)} = 0$, then there exists a unique solution to* (4).

Thus, we directly give the following assumption for simplicity.

Assumption 1. *The assumptions or conditions given by Theorem 1 are fulfilled throughout the current paper.*

2.1 Economic Maturity

It is assumed that the economy consists of $L(t)$ identical individuals in period t, each of whom possesses perfect foresight. We thus suppose that there is a representative agent with the following objective function:

$$\mathbb{E}_{t_0}\left[\int_{t_0}^{\tau} e^{-\rho(t-t_0)} U_1\left((1 - s(k(t))) f(k(t))\right) dt + e^{-\rho(\tau-t_0)} U_2\left(f(k(\tau))\right)\right], \tag{5}$$

where $\mathbb{E}_{t_0}$ denotes the expectation operator depending on $\mathcal{F}_{t_0}$ with $t_0 \geq 0$, $0 < \rho < 1$ represents the discount factor, $\tau \equiv \tau(\omega) \in \mathcal{T} \equiv \{\mathcal{F}\text{-stopping times}\}$ for $\omega \in \Omega$, and $U_1(\cdot)$, $U_2(\cdot)$ are strictly increasing, strictly concave instantaneous utility functions of per capita consumption and per capita output, respectively, with the well-known Inada conditions satisfied.

It is easily seen that the criterion defined by (5) is widely used in existing literature, including the macroeconomic studies. Nevertheless, $\tau \equiv \tau(\omega)$ is usually pre-specified and deterministic, e.g., $\tau(\omega) \equiv T > 0$ for all $\omega \in \Omega$ and any exogenously given constant $0 < T \leq \infty$. Noting that τ truly implies interesting and important economic implications in accordance to Kurz (1965) and Dai (2012), we will extend Kurz's work by introducing nature or social planner into the present macroeconomic model. The nature will actually choose an admissible value $\tau^* \equiv \tau^*(\omega)$ so that (5) is maximized. Formally, we give the following definition.

Definition 1. The *stochastic dynamic game* Γ between the nature and the representative agent proceeds according to the following timing structure:

Stage 1: Taking the remaining parameters as given, the nature will determine an optimal stopping time $\tau^*(\omega) \in \mathcal{T}$ such that the criterion in (5) is maximized subject to constraint (4) (i.e., this is essentially an optimal stopping problem).

Stage 2: Given the knowledge of the game structure as well as $\tau = \tau^*(\omega) \in \mathcal{T}$, the representative agent chooses an optimal savings strategy $s^*(k(t), \tau^* - t_0)$ such that the criterion defined in (5) is maximized subject to constraint (4).

Then, following the classical Backward Induction Principle, we formulate:

Problem 1. The representative agent will find a savings policy $s^*(k(t), \tau - t_0)$ so as to

$$\max \mathbb{E}_{t_0}\left[\int_{t_0}^{\tau} e^{-\rho(t-t_0)} U_1\left((1 - s(k(t))) f(k(t))\right) dt + e^{-\rho(\tau-t_0)} U_2\left(f(k(\tau))\right)\right]$$

subject to the SDE of capital-labor ratio in (4), for $\forall \tau \in \mathcal{T}$.

If Problem 1, the modified Ramsey (1928) problem, has a solution, we obtain the optimal path of capital-labor ratio as follows:

$$dk(t) = \left[s^*(k(t), \tau - t_0) f(k(t)) - \left(\delta + n - \sigma^2\right) k(t)\right] dt - \sigma k(t) dB(t). \tag{6}$$

And we put:

Problem 2. The optimization problem facing the nature is to find a stopping rule $\tau^*(\omega) \in \mathcal{T}$ so as to

$$\sup \mathbb{E}_{t_0}\left[\int_{t_0}^{\tau} e^{-\rho(t-t_0)} U_1\left((1 - s^*(k(t), \tau - t_0)) f(k(t))\right) dt + e^{-\rho(\tau-t_0)} U_2\left(f(k(\tau))\right)\right]$$

subject to the SDE of capital-labor ratio given by (6).

Remark 1.

(i) It is especially worth emphasizing that Problem 2 can also be modified by focusing entirely upon the final state as that of Radner (1961) and Dai (2012). That is, the criterion of preference facing the nature is given by

$$\mathbb{E}_{t_0}\left[e^{-\rho(\tau-t_0)} U_2\left(f(k(\tau))\right)\right],$$

which, in general, will result in a new turnpike. Nevertheless, we argue that similar turnpike theorems can be established for the new turnpike.

(ii) In particular, one may notice certain similarity of the present approach to the literature studying endogenous lifetime or endogenous longevity in growth models (see Chakraborty 2004; de la Croix and Ponthiere 2010, and among others), obvious differences, nevertheless, exist between the both, especially when referring to economic intuition and implications behind formal models. Existing studies focus on OLG models and health-investment behaviors while the current exploration emphasizes issues of macroeconomic development, namely, the characterization of economic maturity for underdeveloped economies and the corresponding characteristics of their optimal capital-accumulation paths.

(iii) The *maximum sustainable capital-labor ratio* corresponding to the state of economic maturity as well as the *minimum-time needed to economic maturity* is endogenously determined by using stochastic optimal stopping theory that is widely applied in mathematical finance. However, in Kurz's (1965) study, the target or the maximum sustainable level of terminal path capital-labor ratio is exogenously specified, and the corresponding minimum time problem is expressed as: For any given initial capital-labor ratio, to chose strategies so that the prescribed target can be reached as soon as possible. As a result, the major contribution of the present approach can be expressed as follows: first, we endogenously determine the terminal path of capital-labor ratio as well as the minimum time needed to reach economic maturity; second, we maximize the welfare of the representative agent in solving the minimum-time objective problem.

(iv) It follows from the specification of Problem 2 that we focus on the episode before reaching economic maturity as concentrated in Kurz (1965), Samuelson (1965) and Cass (1966). Put it differently, the present framework is suitable for the studies concerning underdeveloped economies.

Thus, if Problem 2 has a solution, we get the optimal stopping time $\tau^*(\omega) \in \mathcal{T}$, which actually defines the *minimum time needed toward economic maturity*. Also, $(\tau^*(\omega), s^*(k(t), \tau^*(\omega) - t_0))$ forms the sub-game perfect Nash equilibrium (SPNE) of the stochastic dynamic game Γ given by Definition 1.

Remark 2. It is especially worth mentioning that we define the standard of economic maturity from the perspective of economic welfare, which is of course reasonable in the current model economy. Notice that the state of economic maturity for any given economy should imply a peak state that yields the highest level of economic welfare,[1] we argue that the minimum time needed to economic maturity is well-defined.

Finally, noting that we do not focus on the endogenous savings behavior of the representative agent and also the explicit formula of the minimum time needed to economic maturity in the current paper, we can directly put:

Assumption 2. *Let Problem 1 and Problem 2 be solvable, i.e., we can find at least one optimal savings policy* $s^*(k(t), \tau^*(\omega) - t_0)$ *and at least one minimum time* $\tau^*(\omega) \in \mathcal{T}$ *needed toward economic maturity. Moreover, let there exist a constant capital-labor ratio* $0 < k^* < \infty$ *such that the optimal stopping rule is characterized by* $\tau^*(\omega) \equiv \inf\{t \geq 0; k(t) = k^*\} < \infty$ *a.s.-*$\mathbb{P}$.

[1] We, of course, admit that there are many other standards that can characterize the state of economic maturity. Nevertheless, we argue that *economic welfare* will always be the appropriate choice when noting that the major goal of economic growth and economic development is to improve the economic welfare of the people for any modern economies. And in order to make things easier and tractable, we focus on the highest level of economic welfare, and this assumption is, however, without any loss of generality in the underlying economy.

Remark 3.

(i) In fact, Problem 1 can be solved by employing stochastic dynamic programming, and Merton (1975) proved the existence of optimal savings policy in a quite similar case. On the other hand, Problem 2 can also be solved under certain conditions, and one can refer to Karatzas and Wang (2001), Jeanblanc et al. (2004), and Øksendal and Sulem (2005) for more details. The major goal of the present exploration is to confirm that k^* defines a robust turnpike, which is certainly deduced by economic maturity based on the above constructions.

(ii) Assumption 2 ensures the existence of turnpikes from the viewpoint of mathematical techniques. We, however, emphasize that the existence can be taken for granted in reality. In other words, for any developed economy, it experienced the state of economic maturity in history. Thus, the existence of the state of economic maturity is relatively easily ensured in reality.

3. Turnpike theorems

Now, based on Assumption 2, we get

$$\begin{aligned} dk(t) &= \left[s^*\left(k(t), \tau^* - t_0\right) f\left(k(t)\right) - \left(\delta + n - \sigma^2\right) k(t)\right] dt - \sigma k(t) dB(t) \\ &\equiv \varphi\left(k(t)\right) dt + \psi\left(k(t)\right) dB(t) \end{aligned} \tag{7}$$

subject to $k(0) \equiv k_0 > 0$, a deterministic constant. And also,

$$\tau^*(\omega) \equiv \inf\{t \geq 0; k(t) = k^*\} < \infty \text{ a.s.-}\mathbb{P} \tag{8}$$

for some endogenously given constant $0 < k^* < \infty$. We are to show that k^* exhibits turnpike property providing the above assumptions.

Theorem 2 (Asymptotic Turnpike Theorem).[2] *Provided the SDE of capital-labor ratio defined in (7) and the minimum time needed to economic maturity given by (8), then we always get that $k(t)$ converges in $L^1(\mathbb{P})$ and the corresponding limit belongs to $L^1(\mathbb{P})$. In particular, if we have $\varphi\left(k(t)\right) = 0$ a.s.-$\mathbb{P}$, i.e., $s^*\left(k(t), \tau^* - t_0\right) f\left(k(t)\right) = (\delta + n - \sigma^2)k(t)$ a.s.-$\mathbb{P}$, it uniformly converges to k^* a.s.-$\mathbb{P}$, or equivalently,*

$$\lim_{t' \to \infty} \mathbb{P}\left(\bigcup_{t=t'}^{\infty} [|k(t) - k^*| \geq \varepsilon]\right) = 0, \quad \forall \varepsilon > 0.$$

Proof. See Appendix A.

Remark 4. By applying supermartingale property to confirm the corresponding convergence, Joshi (1997) studies the turnpike theory in a stochastic aggregate growth

[2] This proof brings the idea from Dai (2012). Our turnpike theorems satisfy the classical characteristics, i.e., any optimal paths stay within a small neighborhood of the turnpike almost all the time and the turnpike is independent of initial conditions (see McKenzie 1976; Yano 1984; Dai 2014c).

model in which stochastic environments as independent variables are directly and exogenously incorporated into the production function. However, one may easily tell the difference between Joshi's method and our proof. Moreover, it is argued that the essential requirement in Theorem 2 can be easily met thanks to the volatility term σ.

However, if $\varphi(k(t)) \neq 0$, we can define a new process $\theta(t)$ by

$$\varphi(k(t)) = \theta(t)\psi(k(t))$$

for almost all (hereafter a.a.) $(t,\omega) \in [0,T] \times \Omega$. Then we can put

$$Z(t) \equiv \exp\left\{ -\int_0^t \theta(s)dB(s) - \frac{1}{2}\int_0^t \theta^2(s)ds \right\}.$$

Define a new measure $\mathbb{Q}$ on $\mathcal{F}_T$ by

$$d\mathbb{Q}(\omega) = Z(T)d\mathbb{P}(\omega),$$

i.e., $Z(T)$ is the Radon-Nikodym derivative. In what follows, we first introduce the following assumption:

Assumption 3. *At least one of the following two conditions holds:*

(i) $\mathbb{E}[Z(T)] = 1$.

(ii) *The following Novikov Condition holds, i.e.,*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta^2(t)dt\right)\right] < \infty \ \ \textit{for} \ \ 0 \leq T < \infty.$$

Thus, based upon Assumption 3 and according to the Girsanov Theorem, we get that $\mathbb{Q}$ is a probability measure on $\mathcal{F}_T$, $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $k(t)$ is a martingale w. r. t. $\mathbb{Q}$ on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$. Using Girsanov Theorem again, we claim that the process

$$\hat{B}(t) \equiv \int_0^t \theta(s)ds + B(t), \quad \forall t \in [0,T]$$

is a Brownian motion w. r. t. $\mathbb{Q}$ with $\hat{B}(0) = B(0) = 0$ a.s., and expressed in terms of $\hat{B}(t)$ we can get

$$dk(t) = \psi(k(t))\,d\hat{B}(t) \tag{9}$$

subject to $k(0) \equiv k_0 > 0$, a deterministic constant.

Next, based on (9) and similar to (8), we, by slightly modifying Assumption 2, have

$$\hat{\tau}^*(\omega) \equiv \inf\left\{t \geq 0; k(t) = \hat{k}^*\right\} < \infty \ \text{ a.s.-}\mathbb{Q} \tag{10}$$

for some endogenously determined $0 < \hat{k}^* < \infty$.

Here, the operation of changing probability measure is mainly technically motivated. These turnpike properties rely on a martingale feature of the equilibrium path of capital accumulation. Before changing the probability measure, we just focus attention on a special case where the equilibrium path of capital accumulation is already a martingale process. If we relax the assumption used in such a special case, we then need to change the probability measure to obtain a martingale process by making use of the well-known Girsanov Theorem. Therefore, employing similar proof as that of Theorem 2, we can establish:

Theorem 3 (**Asymptotic Turnpike Theorem**). *Provided the SDE of capital-labor ratio defined in (9) and the minimum time needed to economic maturity given by (10), then we always get that $k(t)$ converges in $L^1(\mathbb{Q})$ and the corresponding limit belongs to $L^1(\mathbb{Q})$. Specifically, it uniformly converges to $\hat{k}^*$ a.s.-$\mathbb{Q}$, or equivalently,*

$$\lim_{t' \to \infty} \mathbb{Q}\left(\bigcup_{t=t'}^{\infty} \left[\left|k(t) - \hat{k}^*\right| \geq \varepsilon\right]\right) = 0, \quad \forall \varepsilon > 0.$$

Now, we proceed to prove the neighborhood turnpike theorem. We do this by first giving the following assumption.

Assumption 4. *Let $k(t) \in \Re_{++} \equiv [0, \infty]$, which is the one point compactification of $\Re$ at infinity with the induced topology, $\forall t \geq 0$. Also, there exists a unique invariant Borel probability measure π defined on $\Re_{++}$ such that $\pi\left[bd\left(\Re_{++}\right)\right] \equiv \pi\left[\{0\} \bigcup \{+\infty\}\right] = 0$, where $bd\left(\Re_{++}\right)$ denotes the boundary of $\Re_{++}$. We particularly denote by $\hat{\pi}$ the Borel probability measure corresponding to the SDE defined in (9).*

Remark 5. Mirman (1972) and Dai (2014c) construct a one-sector growth model with uncertain technology, i.e., random variables, which are assumed to be independent and identically distributed, are directly introduced into the neoclassical production function, thereby resulting in a discrete-time Markov process of capital stock. Specifically, Mirman defines Borel probability measure on the Borel sets of non-negative real line by using the corresponding probability transition function of the Markov process. Moreover, Theorem 2.1 of Mirman confirms that there exists a stationary probability measure that has no mass at either zero or infinity. In contrast, the present paper constructs continuous time Markov process of capital-labor ratio. Nonetheless, one can still prove that there exists a unique invariant Borel probability measure satisfying the requirements of Assumption 4 under relatively weak conditions. For more details, one may refer to Theorem 2.1 of Imhof (2005), Theorem 3.1 of Benaïm et al. (2008) and Theorem 5 of Schreiber et al (2011). The present paper omits the corresponding proof to economize on the space.

As a consequence, the following theorem is derived.

Theorem 4 (**Neighborhood Turnpike Theorem**).[3] *Provided assumptions of Theorem 2 are fulfilled and also Assumption 4 holds, we can get that there exists a constant*

[3] This proof brings the method employed by Imhof (2005) and Dai (2012).

$\Sigma > 0$ *such that for* $\forall\alpha > 0$ *with* $\alpha > \Sigma$,

(i) $\mathbb{E}\left[\tau_{\overline{B}_{\alpha}(k^*)}(\omega)\right] \leq \dfrac{dist\,(k_0,k^*)}{\alpha-\Sigma}$,

(ii) $\pi\left[\overline{B}_{\alpha}(k^*)\right] \geq 1-\frac{\Sigma}{\alpha} \equiv 1-\varepsilon$,

where

$$\begin{aligned} B_{\alpha}(k^*) &\equiv \{k(t)\in\Re_{++};|k(t)-k^*|<\alpha, t\geq 0\}, \\ \tau_{\overline{B}_{\alpha}(k^*)}(\omega) &\equiv \inf\left\{t\geq 0; k(t)\in\overline{B}_{\alpha}(k^*)\equiv clB_{\alpha}(k^*)\right\}, \textit{ and} \\ dist\,(k_0,k^*) &\equiv k^*\log\left(k^*/k_0\right) \end{aligned}$$

for $(k^*>)k_0\equiv k(0)>0$.

Proof. See Appendix B.

In particular, this result just considers the case with $k^* > k_0$. Definitely, we can obtain similar result for the case with $k^* < k_0$ through redefining the distance function as $dist\,(k_0,k^*) \equiv k_0\log\left(k_0/k^*\right)$.

Similarly, we derive the following theorem.

Theorem 5 (**Neighborhood Turnpike Theorem**). *Provided assumptions of Theorem 3 are fulfilled and also Assumption 4 holds, we can get that there exists a constant* $\hat{\Sigma} > 0$ *such that for* $\forall\hat{\alpha} > 0$ *with* $\hat{\alpha} > \hat{\Sigma}$,

(i) $\mathbb{E}^{\mathbb{Q}}\left[\hat{\tau}_{\overline{B}_{\hat{\alpha}}(\hat{k}^*)}(\omega)\right] \leq \dfrac{dist\left(k_0,\hat{k}^*\right)}{\hat{\alpha}-\hat{\Sigma}}$,

(ii) $\hat{\pi}\left[\overline{B}_{\hat{\alpha}}(\hat{k}^*)\right] \geq 1-\frac{\hat{\Sigma}}{\hat{\alpha}} \equiv 1-\hat{\varepsilon}$,

where

$$\begin{aligned} B_{\hat{\alpha}}\left(\hat{k}^*\right) &\equiv \left\{k(t)\in\Re_{++};\left|k(t)-\hat{k}^*\right|<\hat{\alpha}, t\geq 0\right\}, \\ \hat{\tau}_{\overline{B}_{\hat{\alpha}}(\hat{k}^*)}(\omega) &\equiv \inf\left\{t\geq 0; k(t)\in\overline{B}_{\hat{\alpha}}(\hat{k}^*)\equiv clB_{\hat{\alpha}}\left(\hat{k}^*\right)\right\}, \textit{ and} \\ dist\left(k_0,\hat{k}^*\right) &\equiv \hat{k}^*\log\left(\hat{k}^*/k_0\right) \end{aligned}$$

for $(\hat{k}^*>)k_0\equiv k(0)>0$.

Remark 6. Theorem 4 shows that the Borel probability measure π will place nearly all mass close to the turnpike k^*. Similarly, Theorem 5 reveals that the corresponding probability distribution $\hat{\pi}$ will place almost all mass close to the new turnpike $\hat{k}^*$. Indeed, Theorems 4 and 5 demonstrate the turnpike property from both time dimension and space dimension, i.e., in the sense of Markov time as well as invariant probability distribution, which of course will provide us with a much more complete characterization of the neighborhood turnpike property when compared with existing studies (see McKenzie 1976; Bewley 1982; Yano 1984; Dai 2014c).

What's the potential application of our theoretical result? It seems hard to see any direct application of our abstract assertion, we, however, will offer the following implication to reveal the potential practical-value of our theoretical argument. The finding in Theorem 5 leads us to a much more comprehensive philosophy when we are motivated to comparatively analyze capital accumulation within different economic systems. For example, for two economies with different levels of economic maturity, e.g., the first one is relatively higher than the second one. Hence, we usually claim that the first one will economically dominate the second one. Nonetheless, our result argues that this conclusion is really hasty and hence may not be comprehensive, and it even does not make any sense. Why? When we attempt to evaluate the potential of capital accumulation for different economies, we should simultaneously consider efficiency from the time aspect, e.g., the first economy may take 15 years to reach its neighborhood efficiency, whereas the second one only takes 5 years. In other words, Theorem 5 confirms that both the height of our goal and the speed leading toward our goal are equivalently crucial from the perspective of evaluating economic efficiency.

Not only that, we are encouraged to add the following comment for Theorem 5. It is worthwhile indicating that there exists an intriguing relation between our major result and the concept of flexibility. In fact, we understand the concept of flexibility under the current background like this: It captures the dynamic trade-off between evaluation accuracy and sustainable economic incentive. To be exact, the selected scope or radius of the given neighborhood completely determined by the exogenous parameter $\hat{\alpha}$ reflects the underlying flexibility of the evaluation mechanism proposed by Theorem 5. In particular, if we are to pursue a relatively high goal of economic maturity, then we can properly extend the given neighborhood; symmetrically, if the goal is relatively low, then we can proportionally narrow the neighborhood. Therefore, we are kept in a subtle balance between the *evaluation accuracy* and the *sustainable economic incentive*. As is broadly recognized, accuracy is important because it reveals useful information of the real macroeconomic process and meanwhile avoids any unnecessary overconfidence, while economic incentive is sustainable only when there are external encouragements from real accomplishments. In sum, policy makers should carefully sustain such a balance. It, therefore, can be regarded as an insightful lesson policy makers might have learned from our theoretical model.

4. Robustness

Before establishing the formal assertion, we first give the following definition.

Definition 2 (**Robust Turnpike**). For a turnpike of any given equilibrium path of capital accumulation, we call it a *robust turnpike* if any perturbed equilibrium path of capital accumulation asymptotically converges to it as the perturbation term (or vector) approaches zero (or a zero vector).

It follows from (7) that

$$dk(t) = \varphi(k(t))\,dt + \psi(k(t))\,dB(t) \equiv k(t)\varphi_0(k(t))\,dt + k(t)\psi_0(k(t))\,dB(t). \tag{11}$$

Now, we introduce the following SDE:

$$d\tilde{k}(t) = \tilde{\varphi}\left(\tilde{k}(t)\right) dt + \tilde{\psi}\left(\tilde{k}(t)\right) dB(t) \equiv \tilde{k}(t)\tilde{\varphi}_0\left(\tilde{k}(t)\right) dt + \tilde{k}(t)\tilde{\psi}_0\left(\tilde{k}(t)\right) dB(t), \quad (12)$$

where we have provided the following assumption.

Assumption 5. *For any $\xi > 0$,*

$$\sup_{k,\tilde{k}>0}\left|\varphi_0(k) - \tilde{\varphi}_0(\tilde{k})\right| \vee \sup_{k,\tilde{k}>0}\left|\psi_0(k) - \tilde{\psi}_0(\tilde{k})\right| \leq \xi.$$

That is to say, (12) defines the ξ- perturbation of (11).

Moreover, we need the following assumption.

Assumption 6. *There exist constants $\phi, \tilde{\phi}$ and $\phi_0 < \infty$ such that*

$$|\varphi(k)k| \vee |\psi(k)|^2 \leq \phi\,|k|^2, \quad \left|\tilde{\varphi}(\tilde{k})\tilde{k}\right| \vee \left|\tilde{\psi}(\tilde{k})\right|^2 \leq \tilde{\phi}\,\left|\tilde{k}\right|^2,$$

$$\textit{and} \quad \sup_{k>0}|\varphi_0(k))|^2 \vee \sup_{k>0}|\psi_0(k)|^2 \leq \phi_0 \quad \textit{for } \forall k > 0 \textit{ and } \forall\tilde{k} > 0.$$

Remark 7. One can easily find that Assumption 6 is truly reasonable thanks to Assumption 1. Assumption 6 is indeed without any loss of generality and is just for the sake of convenience in the following proofs.

Lemma 1. *Provided the above assumptions hold, there exist constants $e(k_0, p, T) < \infty$ and $\tilde{e}(k_0, p, T) < \infty$ such that*

(i) $\mathbb{E}\left[\sup_{0\leq t\leq T}|k(t)|^p\right] \leq e(k_0, p, T)$, *and*

(ii) $\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\tilde{k}(t)\right|^p\right] \leq \tilde{e}(k_0, p, T)$

for $k(0) = \tilde{k}(0) = k_0 > 0, \forall T > 0, \forall p \in \mathbb{N}$ (the set of natural numbers), and $p \geq 2$.

Proof. See Appendix C.

Specifically, even if we do not rely on the above assumptions, we can still get the following result:

Lemma 2. *If both $k(t)$ and $\tilde{k}(t)$ are martingales w. r. t. $\mathbb{P}$, then there exist constants $\eta < \infty$ and $\tilde{\eta} < \infty$ such that*

(i) $\mathbb{E}\left[\lim_{T\to\infty}\sup_{0\leq t\leq T}|k(t)|^2\right] < \eta$, *and*

(ii) $\mathbb{E}\left[\lim_{T\to\infty}\sup_{0\leq t\leq T}\left|\tilde{k}(t)\right|^2\right] < \tilde{\eta}$

for $k(0) = \tilde{k}(0) = k_0 > 0$.

Proof. See Appendix D.

Now, we can derive the following proposition.

Proposition 1. *Provided the above assumptions hold and suppose that* $k(0) = \tilde{k}(0) = k_0 > 0$*, then we get*

$$\mathbb{E}\left[\lim_{T\to\infty} \sup_{0\leq t\leq T} \left|k(t) - \tilde{k}(t)\right|^2\right] \to 0$$

as $\xi \to 0$.

Proof. See Appendix E.

Remark 8. It should be pointed out that in the proof of Proposition 1 we have implicitly used the following assumptions, namely, the speed for ξ to approach zero is much faster than the speed for time T to approach infinity and also $0 \times \infty \equiv 0$. Moreover, we can get the same conclusion by taking the limit as $\xi \to 0$ first and then taking the limit along $T \to \infty$.

Accordingly, the following theorem is established.

Theorem 6 (Robust Turnpike). *Provided assumptions of Theorems 2 and 4 are fulfilled,* k^* *is a robust turnpike.*

Proof. To prove the robustness, one just needs to combine Theorem 2 with Proposition 1 (or combine Theorem 4 with Proposition 1) and also use the following fact:

$$\left|\tilde{k}(t) - k^*\right|^2 = \left|\tilde{k}(t) - k(t) + k(t) - k^*\right|^2 \leq 2\left[\left|\tilde{k}(t) - k(t)\right|^2 + |k(t) - k^*|^2\right].$$

Thus, we leave the details to the interested reader.

Similarly, one can also arrive at the following result.

Theorem 7 (Robust Turnpike). *Provided assumptions of Theorems 3 and 5 are fulfilled,* $\hat{k}^*$ *is a robust turnpike.*

Remark 9. Theorems 6 and 7 have confirmed the asymptotic stability of turnpikes k^* and $\hat{k}^*$, respectively. To summarize, by noticing that our theorems show that the optimal path of capital accumulation will *robustly* converge to the corresponding turnpike *in the sense of uniform topology*, we argue that the current study indeed extends existing turnpike theorems (see Scheinkman 1976; McKenzie 1983; Yano 1998; Dai 2014c) to much stronger cases. This would be regarded as one contribution of the present paper.

5. Concluding remarks

In the current exploration, we are encouraged to study the economic maturity of a one-sector neoclassical model with stochastic growth. To the best of our knowledge, we, for the first time, provide a relatively complete characterization of the minimum time needed to economic maturity for any underdeveloped economy and further show that the corresponding capital-labor ratio exhibits both asymptotic turnpike property and neighborhood turnpike property under reasonable conditions. In other words, the optimal path of capital accumulation (or the equilibrium path of capital accumulation) will uniformly and robustly converge to the turnpike capital-labor ratio or will spend almost all the time staying in any given neighborhood of the turnpike capital-labor ratio under weak conditions and in a persistently non-stationary environment.

Noting that we assume very general forms of preference for the representative agent and production technology for the firm, one can apply the present framework to study different macroeconomic models with stochastic economic growth. Indeed, the present basic model can be naturally extended to other cases, including multi-sector models, heterogeneous-agent models or dynamic general equilibrium models (e.g., Bewley 1982; Yano 1984, and among others). Finally, as an interesting conjecture, the present framework can be extended to include multiple priors via applying the theory developed by Riedel (2009).

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Appendix

A. Proof of Theorem 2

Put $\varphi(k(t)) = 0$ in (7), then we find that $k(t)$ will be a martingale w. r. t. $\mathbb{P}$. Thus, by using Doob's Martingale Inequality, we obtain

$$\mathbb{P}\left(\sup_{0\le t\le T} |k(t)| \ge \lambda\right) \le \frac{1}{\lambda}\mathbb{E}\left[|k(T)|\right] = \frac{k_0}{\lambda} \text{ for } \forall\lambda > 0,\ \forall T > 0. \tag{A1}$$

Without loss of generality, we put $\lambda = 2^m$ for $m \in \mathbb{N}$, then

$$\mathbb{P}\left(\sup_{0\le t\le T} |k(t)| \ge 2^m\right) \le \frac{1}{2^m}k_0 \text{ for } \forall m \in \mathbb{N} \text{ and } \forall T > 0.$$

Using the well-known Borel-Cantelli Lemma, we arrive at

$$\mathbb{P}\left(\sup_{0\le t\le T} |k(t)| \ge 2^m i.m.m\right) = 0 \text{ for } \forall T > 0,$$

in which $i.m.m$ represents "infinitely many m." So, for a.a. (almost all) $\omega \in \Omega$, there exists $\overline{m}(\omega) \in \mathbb{N}$ such that

$$\sup_{0\le t\le T} |k(t)| < 2^m \text{ a.s. (almost surely) for } m \ge \overline{m}(\omega) \text{ and } \forall T > 0,$$

hence,

$$\lim_{T\to\infty} \sup_{0\le t\le T} |k(t)| \le 2^m \text{ a.s. for } m \ge \overline{m}(\omega).$$

Consequently, $k(t) = k(t,\omega)$ is uniformly bounded for $t \in [0,T]$, $\forall T > 0$ and a.a. $\omega \in \Omega$. Thus, it is ensured that $k(t) = k(t,\omega)$ converges a.s.-$\mathbb{P}$ and the limit belongs to space $L^1(\mathbb{P})$ thanks to Doob's Martingale Convergence Theorem. Moreover, by applying Kolmogorov's (or Chebyshev's) Inequality, we get

$$\mathbb{P}\left(\sup_{0\leq t\leq T}|k(t)|\geq\lambda\right)\leq\frac{1}{\lambda^2}var\left[|k(T)|\right] \text{ for } \forall 0<\lambda<\infty \text{ and } \forall T>0.$$

It follows from (A1) that

$$\frac{1}{\lambda^2}var\left[|k(T)|\right]\leq\frac{k_0}{\lambda} \Leftrightarrow var\left[|k(T)|\right]\leq\lambda k_0 \text{ for } \forall T>0. \tag{A2}$$

Noting that $var\left[|k(T)|\right]=\mathbb{E}\left[|k(T)|^2\right]-(k_0)^2$ for $\forall T>0$, we get by (A2)

$$\mathbb{E}\left[|k(T)|^2\right]\leq(\lambda+k_0)k_0<\infty \text{ for } \forall 0<\lambda<\infty \text{ and } \forall T>0,$$

which yields

$$\sup_{T\geq 0}\mathbb{E}\left[|k(T)|^2\right]\leq(\lambda+k_0)k_0<\infty.$$

Hence, by applying Doob's Martingale Convergence Theorem again, $k(t)=k(t,\omega)$ converges in $L^1(\mathbb{P})$.

Furthermore, it is easily seen that $k(t)-k^*$ is also a martingale w. r. t. $\mathbb{P}$. Thus, applying the Doob's Martingale Inequality again implies that

$$\mathbb{P}\left(\sup_{0\leq t\leq T}|k(t)-k^*|\geq\varepsilon\right)\leq\frac{1}{\varepsilon}\mathbb{E}\left[|k(T)-k^*|\right] \text{ for } \forall\varepsilon>0 \text{ and } \forall T>0. \tag{A3}$$

Provided that $\tau^*(\omega)\equiv\inf\{t\geq 0;k(t)=k^*\}<\infty$ a.s.-$\mathbb{P}$ given by (8), we see that there exists $\beta>0$ such that the martingale inequality in (A3) still holds for $\forall\tau(\omega)\in B_\beta\left(\tau^*(\omega)\right)\equiv\{\tau(\omega)\in\mathcal{T};|\tau(\omega)-\tau^*(\omega)|\leq\beta\}$ by using Doob's Optional Sampling Theorem. Then, we get that $k(\tau)-k^*$ is uniformly bounded on the compact set $B_\beta\left(\tau^*(\omega)\right)$ by applying Heine-Borel Theorem and Weierstrass Theorem. Therefore, we, without any loss of generality, set up $\beta=2^{-m}$ for $\forall m\in\mathbb{N}$. Employing the continuity of martingale w. r. t. time t for any given $\omega\in\Omega$ and $\forall\tau_m\in B_\beta\left(\tau^*(\omega)\right)\equiv B_{2^{-m}}\left(\tau^*(\omega)\right)$ and using the Lebesgue Dominated Convergence Theorem, we get

$$\limsup_{m\to\infty}\mathbb{P}\left(\sup_{0\leq t\leq\tau_m}|k(t)-k^*|\geq\varepsilon\right)\leq\frac{1}{\varepsilon}\limsup_{m\to\infty}\mathbb{E}\left[|k(\tau_m)-k^*|\right]=0$$

almost surely. And this implies that

$$\limsup_{m\to\infty}\mathbb{P}\left(\sup_{0\leq t\leq\tau_m}|k(t)-k^*|<\varepsilon\right)\geq 1 \text{ a.s.-}\mathbb{P}.$$

Letting $\varepsilon=2^{-m_0}$ for $\forall m_0\in\mathbb{N}$, we get

$$\limsup_{m\to\infty}\mathbb{P}\left(\sup_{0\leq t\leq\tau_m}|k(t)-k^*|<2^{-m_0}\right)=1 \text{ a.s.-}\mathbb{P} \text{ for } \forall m_0\in\mathbb{N}.$$

It follows from Fatou's Lemma that

$$\mathbb{P}\left(\sup_{0\leq t\leq \tau^*(\omega)} |k(t)-k^*| < 2^{-m_0}\right) = 1 \text{ a.s.-}\mathbb{P} \text{ for } \forall m_0 \in \mathbb{N}.$$

Then, applying Borel-Cantelli Lemma again implies that

$$\mathbb{P}\left(\sup_{0\leq t\leq \tau^*(\omega)} |k(t)-k^*| < 2^{-m_0} i.m.m_0\right) = 1,$$

where $i.m.m_0$ stands for "infinitely many m_0." So for a.a. $\omega \in \Omega$, there exists $\overline{m}_0(\omega) \in \mathbb{N}$ such that

$$\sup_{0\leq t\leq \tau^*(\omega)} |k(t)-k^*| < 2^{-m_0} \text{ a.s. for } \forall m_0 \geq \overline{m}_0(\omega).$$

That is,

$$\limsup_{m_0\to\infty} \sup_{0\leq t\leq \tau^*(\omega)} |k(t)-k^*| \leq 0 \text{ a.s.-}\mathbb{P},$$

which yields

$$\limsup_{\tau^*(\omega)\to\infty} \sup_{0\leq t\leq \tau^*(\omega)} |k(t)-k^*| \leq 0 \text{ a.s.-}\mathbb{P}.$$

That is to say,

$$\mathbb{P}\left(\bigcup_{m=1}^{\infty}\bigcap_{t'=0}^{\infty}\bigcup_{t=t'}^{\infty}\left[|k(t)-k^*| \geq \frac{1}{m}\right]\right) = 0.$$

Equivalently, for $\forall m \in \mathbb{N}$, we arrive at

$$\mathbb{P}\left(\bigcap_{t'=0}^{\infty}\bigcup_{t=t'}^{\infty}\left[|k(t)-k^*| \geq \frac{1}{m}\right]\right) = 0,$$

i.e., for $\forall \varepsilon > 0$, we have

$$\lim_{t'\to\infty} \mathbb{P}\left(\bigcup_{t=t'}^{\infty}\left[|k(t)-k^*| \geq \varepsilon\right]\right) = 0,$$

which gives the desired assertion. □

B. Proof of Theorem 4

Given the SDE defined by (7), we can define the following characteristic operator of $k(t)$:

$$\mathcal{A}g(k_0) = \varphi(k_0)\frac{\partial g}{\partial k_0}(k_0) + \frac{1}{2}\psi^2(k_0)\frac{\partial^2 g}{\partial (k_0)^2}(k_0)$$

for any $k_0 \equiv k(0) > 0$. We now define Kullback-Leibler type distance (see Bomze 1991; Imhof 2005) between k_0 and k^* as follows:

$$g(k_0) \equiv dist(k_0, k^*) \equiv k^* \log\left(\frac{k^*}{k_0}\right) \geq 0.$$

Then we get

$$\mathcal{A}g(k_0) = \left[-\varphi(k_0) + \frac{1}{2k_0}\psi^2(k_0)\right]\frac{k^*}{k_0}.$$

By Theorem 2, we find that there exists $T_0 < \infty$ such that

$$\sup_{0 \leq t \leq T} |k(t) - k^*| < \mu \quad \text{for } \forall \mu > 0 \text{ and } \forall T \geq T_0.$$

Thus, we have

$$\mathcal{A}g(k_0) \leq \left[-\varphi(k_0) + \frac{1}{2k_0}\psi^2(k_0)\right]\frac{k^*}{k_0} + \mu - |k(t) - k^*| \equiv \Sigma - |k(t) - k^*|. \tag{B1}$$

Define some new notations:

$$B_\alpha(k^*) \equiv \{k(t) \in \Re_{++}; |k(t) - k^*| < \alpha, t \geq 0\},$$

$$\tilde{\tau}(\omega) \equiv \tau_{\overline{B}_\alpha(k^*)}(\omega) \equiv \inf\left\{t \geq 0; k(t) \in \overline{B}_\alpha(k^*) \equiv clB_\alpha(k^*)\right\},$$

where $\overline{B}_\alpha(k^*)$ denotes the closure of $B_\alpha(k^*)$. Suppose $\alpha > \Sigma$ for every $k(t) \notin \overline{B}_\alpha(k^*)$, i.e., $k(t) \in \overline{B}_\alpha^C(k^*)$, we then get

$$\mathcal{A}g(k_0) \leq -\alpha + \Sigma$$

by using (B1). Thus, by making use of Dynkin's formula,

$$0 \leq \mathbb{E}[g(k(t \wedge \tilde{\tau}))] = g(k_0) + \mathbb{E}\left[\int_0^{t \wedge \tilde{\tau}} \mathcal{A}g(k(s))\,ds\right] \leq g(k_0) + (\Sigma - \alpha)\mathbb{E}[t \wedge \tilde{\tau}(\omega)].$$

Since $t \wedge \tilde{\tau} \uparrow \tilde{\tau}$ as $t \to \infty$, applying Lebesgue Monotone Convergence Theorem results in

$$0 \leq g(k_0) + (\Sigma - \alpha)\mathbb{E}[\tilde{\tau}(\omega)],$$

which produces

$$\mathbb{E}\left[\tau_{\overline{B}_\alpha(k^*)}(\omega)\right] = \mathbb{E}[\tilde{\tau}(\omega)] \leq \frac{g(k_0)}{\alpha - \Sigma} = \frac{dist(k_0, k^*)}{\alpha - \Sigma},$$

as required in (i). Moreover, for some constant $W > g(k_0)$, set up

$$\tau_W = \tau_W(\omega) \equiv \inf\{t \geq 0; g(k(t)) = W\}.$$

Thus, by making use of Dynkin's formula and inequality (B1),

$$\begin{aligned} 0 \le \mathbb{E}\left[g\left(k\left(t\wedge\tau_W\right)\right)\right] &= g\left(k_0\right) + \mathbb{E}\left[\int_0^{t\wedge\tau_W} \mathcal{A}g\left(k(s)\right)ds\right] \\ &\le g\left(k_0\right) - \mathbb{E}\left[\int_0^{t\wedge\tau_W} \left|k(s)-k^*\right|ds\right] + \Sigma\mathbb{E}\left[t\wedge\tau_W(\omega)\right]. \end{aligned}$$

If $W \to \infty$, we get $t \wedge \tau_W(\omega) \to t$. By applying Lebesgue Bounded Convergence Theorem and Levi Lemma, we are led to

$$0 \le g\left(k_0\right) - \mathbb{E}\left[\int_0^t \left|k(s)-k^*\right|ds\right] + \Sigma t,$$

which yields

$$\mathbb{E}\left[\frac{1}{t}\int_0^t \left|k(s)-k^*\right|ds\right] \le \frac{g\left(k_0\right)}{t} + \Sigma.$$

Thus, we have

$$\limsup_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t \left|k(s)-k^*\right|ds\right] \le \Sigma. \tag{B2}$$

If we let $\chi_{\overline{B}_\alpha^C(k^*)}\left(k(t)\right)$ denote the indicator function of set $\overline{B}_\alpha^C(k^*)$, then by (B2) and Assumption 4, we arrive at

$$\begin{aligned} \pi\left[\overline{B}_\alpha^C(k^*)\right] &= \limsup_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t \chi_{\overline{B}_\alpha^C(k^*)}\left(k(s)\right)ds\right] \\ &\le \limsup_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t \frac{\left|k(s)-k^*\right|}{\alpha}ds\right] \le \frac{\Sigma}{\alpha}, \end{aligned}$$

which implies that

$$\pi\left[\overline{B}_\alpha(k^*)\right] \ge 1 - \frac{\Sigma}{\alpha} \equiv 1 - \varepsilon,$$

which gives the desired assertion in (ii). □

C. Proof of Lemma 1

Applying Itô's rule to (11) produces

$$|k(t)|^2 = |k_0|^2 + 2\int_0^t \varphi(k(s))\,k(s)ds + \int_0^t |\psi(k(s))|^2 ds + 2\int_0^t \psi(k(s))\,k(s)dB(s).$$

By using Assumption 6 we get that for $t_1 \in [0,T]$ and some constant $e \equiv e(p,T) < \infty$ (which may be different from line to line throughout this proof),

$$\sup_{0\le t\le t_1} |k(t)|^p \le e\left\{ |k_0|^p + \left[\int_0^{t_1} \phi\,|k(s)|^2 ds\right]^{\frac{p}{2}} + \sup_{0\le t\le t_1}\left|\int_0^t k(s)\psi[k(s)]\,dB(s)\right|^{\frac{p}{2}}\right\}.$$

It follows from Cauchy-Schwarz Inequality (Dai 2014b) that

$$\sup_{0\le t\le t_1} |k(t)|^p \le e\left\{ |k_0|^p + \int_0^{t_1} |k(s)|^p ds + \sup_{0\le t\le t_1}\left|\int_0^t k(s)\psi[k(s)]\,dB(s)\right|^{\frac{p}{2}}\right\}.$$

Taking expectations on both sides and applying the Burkholder-Davis-Gundy Inequality (see Karatzas and Shreve 1991, p. 166) show that

$$\mathbb{E}\left[\sup_{0\le t\le t_1} |k(t)|^p\right] \le e\left\{ |k_0|^p + \int_0^{t_1} \mathbb{E}\left[|k(s)|^p\right] ds + \mathbb{E}\left[\int_0^{t_1} |k(s)|^2\,|\psi(k(s))|^2 ds\right]^{\frac{p}{4}}\right\}. \tag{C1}$$

Now, using the Young Inequality (see Higham et al. 2003), Assumption 6, and Rogers-Hölder Inequality (Dai 2014b) reveals that

$$\mathbb{E}\left[\int_0^{t_1} |k(s)|^2\,|\psi(k(s))|^2 ds\right]^{\frac{p}{4}} \le \mathbb{E}\left[\sup_{0\le t\le t_1} |k(t)|^{\frac{p}{2}} \left(\int_0^{t_1} |\psi(k(s))|^2 ds\right)^{\frac{p}{4}}\right]$$

$$\le \frac{1}{2e}\mathbb{E}\left[\sup_{0\le t\le t_1} |k(t)|^p\right] + \frac{e}{2}\mathbb{E}\left[\int_0^{t_1} |\psi(k(s))|^2 ds\right]^{\frac{p}{2}}$$

$$\le \frac{1}{2e}\mathbb{E}\left[\sup_{0\le t\le t_1} |k(t)|^p\right] + \frac{e}{2}\phi^{\frac{p}{2}}\mathbb{E}\left[\int_0^{t_1} |k(s)|^2 ds\right]^{\frac{p}{2}}$$

$$\leq \frac{1}{2e}\mathbb{E}\left[\sup_{0\leq t\leq t_1}|k(t)|^p\right] + \frac{e}{2}\phi^{\frac{p}{2}}T^{\frac{p-2}{2}}\mathbb{E}\left[\int_0^{t_1}|k(s)|^p\,ds\right].$$

Substituting this into (C1) yields

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|k(t)|^p\right] \leq e\left\{|k_0|^p + \int_0^T \mathbb{E}\left[|k(t)|^p\right]dt\right\}.$$

Thus, by applying the following fact (see Higham et al. 2003):

$$\mathbb{E}\left[|k(t)|^p\right] \leq e\left(1+|k_0|^p\right),$$

we arrive at

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|k(t)|^p\right] \leq e(k_0,p,T) < \infty,$$

which gives the desired result in (i). Noting that the proof of (ii) is quite similar to that of (i), we omit it. And this completes the whole proof. □

D. Proof of Lemma 2

By using Doob's Martingale Inequality, we obtain

$$\mathbb{P}\left(\sup_{0\leq t\leq T}|k(t)|\geq\lambda\right) \leq \frac{1}{\lambda}\mathbb{E}\left[|k(T)|\right] = \frac{k_0}{\lambda} \text{ for } \forall 0<\lambda<\infty \text{ and } \forall T>0. \quad \text{(D1)}$$

Similarly, by applying Kolmogorov's (or Chebyshev's) Inequality, we get

$$\mathbb{P}\left(\sup_{0\leq t\leq T}|k(t)|\geq\lambda\right) \leq \frac{1}{\lambda^2}var\left[|k(T)|\right] \text{ for } \forall 0<\lambda<\infty,\ \forall T>0.$$

It follows from (D1) that

$$\frac{1}{\lambda^2}var\left[|k(T)|\right] \leq \frac{k_0}{\lambda} \;\Leftrightarrow\; var\left[|k(T)|\right] \leq \lambda k_0 \text{ for } \forall T>0. \quad \text{(D2)}$$

Noting that $var\left[|k(T)|\right] = \mathbb{E}\left[|k(T)|^2\right] - (k_0)^2$ for $\forall T>0$, we get by (D2)

$$\mathbb{E}\left[|k(T)|^2\right] \leq (\lambda+k_0)k_0 < \infty \text{ for } \forall 0<\lambda<\infty \text{ and } \forall T>0, \quad \text{(D3)}$$

which implies that $k(t)$ is a square-integrable martingale. We define:

$$\zeta \equiv |k(t)|, \quad \zeta^* \equiv \|k(t)\|_\infty \equiv \sup_{0\leq s\leq t}|k(s)| \text{ and } \|k(t)\|_2 \equiv \left\{\mathbb{E}\left[|k(t)|^2\right]\right\}^{\frac{1}{2}}.$$

Thus, by applying Doob's Martingale Inequality and Fubini Theorem, we arrive at the following result for some constant $N < \infty$:

$$\begin{aligned}
\mathbb{E}\left[\left|\zeta^* \wedge N\right|^2\right] &= 2\int_0^\infty x\mathbb{P}\left(\zeta^*(\omega) \wedge N \geq x\right) dx \\
&\leq 2\int_0^\infty \int_{\{\zeta^*(\omega)\wedge N \geq x\}} \zeta(\omega) d\mathbb{P}(\omega) dx \\
&= 2\int_0^\infty \int_\Omega \zeta(\omega) \chi_{\{\zeta^*(\omega)\wedge N\geq x\}} d\mathbb{P}(\omega) dx \\
&= 2\int_\Omega \zeta(\omega) \int_0^{\zeta^*(\omega)\wedge N} dx d\mathbb{P}(\omega) \\
&= 2\int_\Omega \zeta(\omega) \left(\zeta^*(\omega) \wedge N\right) d\mathbb{P}(\omega) \\
&= 2\mathbb{E}\left[\zeta\left(\zeta^* \wedge N\right)\right].
\end{aligned}$$

It follows from Rogers-Hölder Inequality that

$$\left\|\zeta^* \wedge N\right\|_2^2 = \mathbb{E}\left[\left|\zeta^* \wedge N\right|^2\right] \leq 2\left\|\zeta\right\|_2 \left\|\zeta^* \wedge N\right\|_2,$$

which produces

$$\left\|\zeta^* \wedge N\right\|_2 \leq 2\left\|\zeta\right\|_2.$$

Noting that $\mathbb{E}\left[\left|\zeta^* \wedge N\right|^2\right] \leq N^2 < \infty$, and hence applying Lebesgue Dominated Convergence Theorem leads us to

$$\left\|\zeta^*\right\|_2 \leq 2\left\|\zeta\right\|_2 \quad \Leftrightarrow \quad \left\|\zeta^*\right\|_2^2 \leq 4\left\|\zeta\right\|_2^2,$$

namely,

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|k(s)\right|^2\right] \leq 4\mathbb{E}\left[\left|k(t)\right|^2\right] \leq 4(\lambda + k_0)k_0 < \infty \;\text{ for }\; \forall t \geq 0$$

by using the inequality given by (D3). Accordingly, a canonical application of Lebesgue Monotone Convergence Theorem (or Levi Lemma) gives the required assertion in (i). The proof of (ii) is similar to that of (i), we hence omit it. Therefore, the whole proof is complete. □

E. Proof of Proposition 1

Provided the SDEs defined in (11) and (12), it follows from Lemma 1 that for $\forall 2 \leq p < \infty$ and $\forall T > 0$ there exists some constant $W < \infty$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|k(t)\right|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T}\left|\tilde{k}(t)\right|^p\right] \leq W, \tag{E1}$$

where, by using Assumption 1, we can have

$$k(t) = k_0 + \int_0^t k(s)\varphi_0(k(s))\,ds + \int_0^t k(s)\psi_0(k(s))\,dB(s),$$

$$\tilde{k}(t) = k_0 + \int_0^t \tilde{k}(s)\tilde{\varphi}_0\left(\tilde{k}(s)\right)ds + \int_0^t \tilde{k}(s)\tilde{\psi}_0\left(\tilde{k}(s)\right)dB(s).$$

Moreover, we put $|k(t)| \vee \left|\tilde{k}(t)\right| \leq \overline{W} < \infty$ for $\forall t \geq 0$; otherwise, we just consider $k(t) \wedge \overline{W}$ and $\tilde{k}(t) \wedge \overline{W}$ instead of $k(t)$ and $\tilde{k}(t)$, respectively, to get the desired result by sending $\overline{W}$ to infinity and using Lebesgue Dominated Convergence Theorem. In what follows, we first define the following stopping times:

$$\tau_{\overline{W}} \equiv \inf\left\{t \geq 0; |k(t)| \geq \overline{W}\right\}, \quad \tilde{\tau}_{\overline{W}} \equiv \inf\left\{t \geq 0; \left|\tilde{k}(t)\right| \geq \overline{W}\right\}, \quad \tau^*_{\overline{W}} \equiv \tau_{\overline{W}} \wedge \tilde{\tau}_{\overline{W}}.$$

By using the Young Inequality (see Higham et al. 2003) and for any $R > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\left|k(t)-\tilde{k}(t)\right|^2\right] \\
&= \mathbb{E}\left[\sup_{0\leq t\leq T}\left|k(t)-\tilde{k}(t)\right|^2 \chi_{\left\{\tau_{\overline{W}}>T,\tilde{\tau}_{\overline{W}}>T\right\}}\right] \\
&\quad + \mathbb{E}\left[\sup_{0\leq t\leq T}\left|k(t)-\tilde{k}(t)\right|^2 \chi_{\left\{\tau_{\overline{W}}\leq T, or \tilde{\tau}_{\overline{W}}\leq T\right\}}\right] \\
&\leq \mathbb{E}\left[\sup_{0\leq t\leq T}\left|k\left(t\wedge\tau^*_{\overline{W}}\right)-\tilde{k}\left(t\wedge\tau^*_{\overline{W}}\right)\right|^2 \chi_{\left\{\tau^*_{\overline{W}}>T\right\}}\right] \\
&\quad + \tfrac{2R}{p}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|k(t)-\tilde{k}(t)\right|^p\right] + \frac{1-\frac{2}{p}}{R^{\frac{2}{p-2}}}\mathbb{P}\left(\tau_{\overline{W}}\leq T, or \tilde{\tau}_{\overline{W}}\leq T\right).
\end{aligned} \tag{E2}$$

It follows from (E1) that

$$\mathbb{P}\left(\tau_{\overline{W}} \leq T\right) = \mathbb{E}\left[\chi_{\left\{\tau_{\overline{W}}\leq T\right\}}\frac{\left|k\left(\tau_{\overline{W}}\right)\right|^p}{\overline{W}^p}\right] \leq \frac{1}{\overline{W}^p}\mathbb{E}\left[\sup_{0\leq t\leq T}|k(t)|^p\right] \leq \frac{W}{\overline{W}^p}.$$

Similarly, one can get $\mathbb{P}\left(\tilde{\tau}_{\overline{W}} \leq T\right) \leq W/\overline{W}^p$. So,

$$\mathbb{P}\left(\tau_{\overline{W}} \leq T, or \tilde{\tau}_{\overline{W}} \leq T\right) \leq \mathbb{P}\left(\tau_{\overline{W}} \leq T\right) + \mathbb{P}\left(\tilde{\tau}_{\overline{W}} \leq T\right) \leq \frac{2W}{\overline{W}^p}.$$

Moreover, we obtain by (E1)

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|k(t)-\tilde{k}(t)\right|^p\right] \leq 2^{p-1}\mathbb{E}\left[\sup_{0\leq t\leq T}\left(|k(t)|^p + \left|\tilde{k}(t)\right|^p\right)\right] \leq 2^p W.$$

Hence, (E2) becomes

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\left|k(t)-\tilde{k}(t)\right|^2\right] \\
&\quad \leq \mathbb{E}\left[\sup_{0\leq t\leq T}\left|k\left(t\wedge\tau^*_{\overline{W}}\right)-\tilde{k}\left(t\wedge\tau^*_{\overline{W}}\right)\right|^2\right] + \frac{2^{p+1}RW}{p} + \frac{2(p-2)W}{pR^{\frac{2}{p-2}}\overline{W}^p}.
\end{aligned} \tag{E3}$$

By making use of Cauchy-Bunyakovsky-Schwarz Inequality, we get

$$\begin{aligned}
\left|k\left(t\wedge\tau_W^*\right)-\tilde{k}\left(t\wedge\tau_W^*\right)\right|^2 &= \left|\int_0^{t\wedge\tau_W^*}\left[k(s)\varphi_0\left(k(s)\right)-\tilde{k}(s)\tilde{\varphi}_0\left(\tilde{k}(s)\right)\right]ds\right.\\
&\quad\left.+\int_0^{t\wedge\tau_W^*}\left[k(s)\psi_0\left(k(s)\right)-\tilde{k}(s)\tilde{\psi}_0\left(\tilde{k}(s)\right)\right]dB(s)\right|^2\\
&\leq 2\left\{T\int_0^{t\wedge\tau_W^*}\left[\left|k(s)\varphi_0\left(k(s)\right)-\tilde{k}(s)\tilde{\varphi}_0\left(\tilde{k}(s)\right)\right|^2\right]ds\right.\\
&\quad\left.+\left|\int_0^{t\wedge\tau_W^*}\left[k(s)\psi_0\left(k(s)\right)-\tilde{k}(s)\tilde{\psi}_0\left(\tilde{k}(s)\right)\right]dB(s)\right|^2\right\}\\
&\leq 4\left\{T\int_0^{t\wedge\tau_W^*}\left[\left|k(s)\varphi_0\left(k(s)\right)-\tilde{k}(s)\varphi_0\left(k(s)\right)\right|^2\right]ds\right.\\
&\quad+T\int_0^{t\wedge\tau_W^*}\left[\left|\tilde{k}(s)\varphi_0\left(k(s)\right)-\tilde{k}(s)\tilde{\varphi}_0\left(\tilde{k}(s)\right)\right|^2\right]ds\\
&\quad\left.+\left|\int_0^{t\wedge\tau_W^*}\left[k(s)\psi_0\left(k(s)\right)-\tilde{k}(s)\tilde{\psi}_0\left(\tilde{k}(s)\right)\right]dB(s)\right|^2\right\}.
\end{aligned}$$

Taking expectations on both sides and using Itô's Isometry (Dai, 2014b), we have for $\forall\tau\leq T$:

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq\tau}\left|k\left(t\wedge\tau_W^*\right)-\tilde{k}\left(t\wedge\tau_W^*\right)\right|^2\right]\\
&\leq 4\left\{T\mathbb{E}\left[\int_0^{t\wedge\tau_W^*}\left|k(s)-\tilde{k}(s)\right|^2\left|\varphi_0\left(k(s)\right)\right|^2ds\right]\right.\\
&\quad+T\mathbb{E}\left[\int_0^{t\wedge\tau_W^*}\left|\tilde{k}(s)\right|^2\left|\varphi_0\left(k(s)\right)-\tilde{\varphi}_0\left(\tilde{k}(s)\right)\right|^2ds\right]\\
&\quad\left.+\mathbb{E}\left[\int_0^{t\wedge\tau_W^*}\left|k(s)\psi_0\left(k(s)\right)-\tilde{k}(s)\tilde{\psi}_0\left(\tilde{k}(s)\right)\right|^2ds\right]\right\}\\
&\leq 8\left\{T\phi_0\mathbb{E}\left[\int_0^{t\wedge\tau_W^*}\left|k(s)-\tilde{k}(s)\right|^2ds\right]+T\xi^2\mathbb{E}\left[\int_0^{t\wedge\tau_W^*}\left|\tilde{k}(s)\right|^2ds\right]\right.\\
&\quad+\mathbb{E}\left[\int_0^{t\wedge\tau_W^*}\left|k(s)\psi_0\left(k(s)\right)-\tilde{k}(s)\psi_0\left(k(s)\right)\right|^2ds\right]\\
&\quad\left.+\mathbb{E}\left[\int_0^{t\wedge\tau_W^*}\left|\tilde{k}(s)\psi_0\left(k(s)\right)-\tilde{k}(s)\tilde{\psi}_0\left(\tilde{k}(s)\right)\right|^2ds\right]\right\}\\
&\leq 8\left\{T\phi_0\mathbb{E}\left[\int_0^{t\wedge\tau_W^*}\left|k(s)-\tilde{k}(s)\right|^2ds\right]+T\xi^2\mathbb{E}\left[\int_0^{t\wedge\tau_W^*}\left|\tilde{k}(s)\right|^2ds\right]\right.\\
&\quad\left.+\phi_0\mathbb{E}\left[\int_0^{t\wedge\tau_W^*}\left|k(s)-\tilde{k}(s)\right|^2ds\right]+\xi^2\mathbb{E}\left[\int_0^{t\wedge\tau_W^*}\left|\tilde{k}(s)\right|^2ds\right]\right\}
\end{aligned}$$

$$\begin{aligned} &= 8\left\{(T+1)\phi_0\mathbb{E}\left[\int_0^{t\wedge\tau^*_{\overline{W}}}\left|k(s)-\tilde{k}(s)\right|^2 ds\right]+(T+1)\xi^2\mathbb{E}\left[\int_0^{t\wedge\tau^*_{\overline{W}}}\left|\tilde{k}(s)\right|^2 ds\right]\right\} \\ &\leq 8\left\{(T+1)\phi_0\int_0^T\mathbb{E}\left[\sup_{0\leq t_0\leq s}\left|k\left(t_0\wedge\tau^*_{\overline{W}}\right)-\tilde{k}\left(t_0\wedge\tau^*_{\overline{W}}\right)\right|^2\right]ds+T(T+1)\overline{W}^2\xi^2\right\}, \end{aligned}$$

where we have used Assumptions 5 and 6. Hence, applying Gronwall's Inequality (see Higham et al. 2003; Dai 2014b) gives rise to

$$\mathbb{E}\left[\sup_{0\leq t\leq\tau}\left|k\left(t\wedge\tau^*_{\overline{W}}\right)-\tilde{k}\left(t\wedge\tau^*_{\overline{W}}\right)\right|^2\right]\leq 8T(T+1)\overline{W}^2\exp\left[8(T+1)\phi_0\right]\xi^2.$$

Inserting this into (E3) leads us to

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|k(t)-\tilde{k}(t)\right|^2\right]\leq 8T(T+1)\overline{W}^2\exp\left[8(T+1)\phi_0\right]\xi^2+\frac{2^{p+1}RW}{p}+\frac{2(p-2)W}{pR^{\frac{2}{p-2}}\overline{W}^p}.$$

Hence, for $\forall\varepsilon>0$, we can choose R and $\overline{W}$ such that

$$\frac{2^{p+1}RW}{p}\leq\frac{\varepsilon}{3}\ \text{ and }\ \frac{2(p-2)W}{pR^{\frac{2}{p-2}}\overline{W}^p}\leq\frac{\varepsilon}{3}.$$

And for any given $T>0$, we put ξ such that

$$8T(T+1)\overline{W}^2\exp\left[8(T+1)\phi_0\right]\xi^2\leq\frac{\varepsilon}{3}.$$

In consequence, for $\forall\varepsilon>0$, we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|k(t)-\tilde{k}(t)\right|^2\right]\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Notice the arbitrariness of ε, we can employ Levi Lemma to produce the desired result. This proof is accordingly complete. □