A NOTE ON GENERALIZED GENERALIZATION

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ABSTRACT

The generalization rules of sequent calculi allow, under some restrictions, to derive a formula $\exists x\varphi$ or $\forall x\varphi$ from a formula $\varphi_x(y)$, i.e. from the formula obtained by substituting a variable y for all free occurrences of x in φ. We introduce modified generalization rules that make it possible to derive $\exists x\varphi$ or $\forall x\varphi$ from $\varphi_x(t)$ even in some cases where t is a complex term. These modified rules were invented in connection with attempts to prove the interpolation theorem for classical predicate logic without equality but with function symbols. This theorem seems (and remains) to be an unresolved case in the literature.

Keywords: generalization; sequent; interpolation.

1 Introduction: interpolation theorems

Interpolation theorem, for one or another logic, is easily stated if the list of logical symbols includes the "nulary connectives" $\top$ and $\bot$ for truth and falsity. Then the interpolation theorem is the claim that if an implication $\varphi \to \psi$ is *valid* (as determined by the semantics of the logic in question), then there exists a formula μ, called *interpolant* of φ and ψ, such that $\varphi \to \mu$ and $\mu \to \psi$ are valid and μ contains only those extralogical symbols that simultaneously occur in both φ and ψ. In predicate logic we first choose a language L, and then *extralogical symbols*, or just *symbols*, are free variables, and function and predicate symbols of L. Interpolation makes sense also in various propositional logics (classical, non-classical, modal). Then extralogical symbols are just atoms. We will give some examples of different logics later in Section 3.

Let $\mathrm{Symb}(\varphi)$ or $\mathrm{Symb}(\Gamma)$ for a formula φ or a set Γ of formulas denote the set of all extralogical symbols in φ or in Γ. Thus an interpolant μ of formulas φ and ψ must satisfy $\mathrm{Symb}(\mu) \subseteq \mathrm{Symb}(\varphi) \cap \mathrm{Symb}(\psi)$. In *propositional logic* $\mathrm{Symb}(..)$ is the set of all atoms in φ or in Γ. In *predicate* logic *with equality* $\mathrm{Symb}(..)$ is the set of all free variables, predicate symbols and function symbols that occur in φ or in Γ. In this case the symbol $=$ has a fixed realization in any structure. It is not considered an extralogical symbol and thus it never appears in a set of the form $\mathrm{Symb}(..)$. Just like connectives and quantifiers, it may occur in an interpolant regardless whether it occurs in the interpolated formulas. In predicate logic *without equality* the symbol $=$ is an extralogical binary symbol with no fixed meaning, and it may occur in an interpolant of φ and ψ only if it is in $\mathrm{Symb}(\varphi) \cap \mathrm{Symb}(\psi)$.

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Example 1.1 Let LO be the conjunction of the axioms of strict linear order, i.e. of the sentences $\forall x\forall y\forall z(R(x,y) \,\&\, R(y,z) \rightarrow R(x,z))$, $\forall x\forall y(R(x,y) \,\vee\, x = y \,\vee\, R(y,x))$ and $\forall x\neg R(x,x)$. Let φ be LO & $\forall x\exists y R(x,y)$ and let ψ be $\exists y(z \neq y)$ where $z \neq y$ is a shorthand for $\neg(z = y)$. In classical predicate logic *with equality* $\mathrm{Symb}(\varphi)$ is $\{R\}$ and $\mathrm{Symb}(\psi)$ is $\{z\}$. The implication $\varphi \rightarrow \psi$ is logically valid and thus one can seek an interpolant μ satisfying $\mathrm{Symb}(\mu) = \emptyset$. It is easy to check that $\mu = \forall x\exists y(x \neq y)$ is as required. In predicate logic *without* equality $\mathrm{Symb}(\varphi)$ is $\{R, =\}$ and $\mathrm{Symb}(\psi)$ is $\{z, =\}$. One can verify that now $\varphi \rightarrow \psi$ *is not* logically valid. However, if we denote $\varphi \,\&\, \forall u\forall v(R(u,v) \,\&\, u = v \rightarrow R(u,u))$ by χ, then $\chi \rightarrow \psi$ is logically valid. Then the same formula $\forall x\exists y(x \neq y)$ works as an interpolant of χ and ψ.

In logic with equality one can use equivalences like $\bot \,\&\, \varphi \equiv \bot$ and $\bot \vee \varphi \equiv \varphi$ and verify that every formula μ is equivalent to $\top$, or to $\bot$, or to a formula ν not containing $\top$ and $\bot$ and satisfying $\mathrm{Symb}(\nu) \subseteq \mathrm{Symb}(\mu)$. Thus if $\top$ and $\bot$ are absent, the interpolation theorem reads: if $\varphi \rightarrow \psi$ is valid, then $\neg\varphi$ is valid, or ψ is valid, or there exists a formula ν such that $\varphi \rightarrow \nu$ and $\nu \rightarrow \psi$ are valid and $\mathrm{Symb}(\nu) \subseteq \mathrm{Symb}(\varphi) \cap \mathrm{Symb}(\psi)$. This explains that we do not see our assumption that $\top$ and $\bot$ are present as a restriction. It just simplifies claims and their proofs. Clearly, $\top$ is equivalent to $\bot \rightarrow \bot$ and $\neg\varphi$ is equivalent to $\varphi \rightarrow \bot$. Thus when discussing logical calculi, we will be able to simplify their definitions using the assumptions that $\top$ and $\neg$ are defined symbols.

Besides (normal) interpolation one can also consider *uniform interpolation*. Let φ be a formula and let S be a set of variables and predicate or function symbols such that $S \subseteq \mathrm{Symb}(\varphi)$. A formula μ is a *right uniform interpolant of* φ *with respect to* S if $\varphi \rightarrow \mu$ is valid (again, as determined by the semantics in question), $\mathrm{Symb}(\mu) \subseteq S$, and $\mu \rightarrow \psi$ is valid for any formula ψ such that $\mathrm{Symb}(\psi) \cap \mathrm{Symb}(\varphi) \subseteq S$ and $\varphi \rightarrow \psi$ is valid. Thus a right uniform interpolant of φ with respect to S can be described as the strongest formula μ that is a consequence of φ and satisfies $\mathrm{Symb}(\mu) \subseteq S$. *Left* uniform interpolant is defined analogously.

Example 1.2 Work in classical predicate logic with equality, let LO be the same conjunction as in Example 1.1 and let again φ be the formula LO & $\forall x\exists y R(x,y)$. We have $\mathrm{Symb}(\varphi) = \{R\}$. Consider a right uniform interpolant μ of φ with respect to $\emptyset$. Then μ must be a sentence in the language $L_0 = \emptyset$, i.e. a sentence built up from equalities of variables using connectives and quantifiers. Let m be the number of quantifiers in μ. A structure for L_0 is just a nonempty set (the structure has a domain and no realizations of symbols). Clearly, every *infinite* structure $\mathcal{A}$ for L_0 has an expansion that is a model of φ. Since $\varphi \rightarrow \mu$ is logically valid, we see that μ is valid in every infinite structure $\mathcal{A}$. However, since μ contains only m occurrences of quantifiers, it is also valid in every structure having at least m elements. This claim is a consequence of the following lemma, which can be proved by outer induction on n and inner induction on the number of logical symbols in ψ. *Let $\mathcal{A}$ and $\mathcal{B}$ be structures for L_0, let $f : \mathcal{A} \rightarrow \mathcal{B}$ be one-to-one, let $\psi(x_1, \ldots, x_k)$ be a formula containing at most n quantifiers, let $a_1, \ldots, a_k$ be elements of $\mathcal{A}$, and assume that $\mathcal{A}$ contains at least n elements different from $a_1, \ldots, a_k$ and $\mathcal{B}$ contains at least n elements different from $f(a_1), \ldots, f(a_k)$. Then $\mathcal{A} \models \varphi[a_1, \ldots, a_k]$ if and only if $\mathcal{B} \models \varphi[f(a_1), \ldots, f(a_k)]$.* Knowing that μ is valid in every structure having at least m elements, we have reached a contradiction: the sentence $\forall x_1 \ldots \forall x_m \exists y(y \neq x_1 \ \& \ \ldots \ \& \ y \neq x_m)$ is a consequence of φ, it is not valid in an m-element structure and thus it is not a consequence of μ, which it should be since μ is a right uniform interpolant. Thus we see that a general theorem stating the existence of right uniform interpolants is not true for classical predicate logic with equality.

First papers about interpolation, containing also some applications, are W. Craig's [Cra57a] and [Cra57b]. Then R. C. Lyndon in [Lyn59a] and [Lyn59b] distinguished positive and negative occurrences of symbols and proved a stronger result: every two formulas φ and ψ have an interpolant μ such that every symbol that appears positively (negatively) in μ also appears positively (negatively) in both φ and ψ. Various variants of Craig's or Lyndon's theorem are often cited as the *Craig–Lyndon interpolation theorem*. Henkin in [Hen63] proved (among other things) that uniform interpolation theorems are true for classical propositional logic. Example 1.2 above is also taken from [Hen63]. Later interpolation became a well-established field of research. Now there exists numerous literature about normal or uniform interpolation for different nonclassical logics, and the proofs involve both semantic and proof-theoretic methods. Some idea about this field can be obtained for example from [Bíl07] and from its list of references. Interesting negative results exist as well: [MOU13] show that the interpolation theorem does not hold for logic of constant domains.

Craig and Lyndon proved the interpolation theorem for classical predicate logic with equality, and also for classical predicate logic without equality but with the following additional restriction: *there are no function symbols of nonzero arity*. Also Takeuti and Buss in [Tak75] and [Bus98] work under the same assumption about function symbols. Craig in [Cra57b] says that "most results of this paper do not hold for first-order predicate calculus with function symbols", but does not give any counterexamples. Thus it seems that the case of logic without equality but with no restriction on function symbols is unresolved.

This paper is motivated by this unresolved case, but we will not be able to give an ultimate answer. In the next section we will mention calculi for classical predicate logic. We will put emphasis on Gentzen-style calculi, and we will define generalized (or enhanced) generalization rules that have been invented during attempts to prove an unrestricted interpolation theorem for classical logic without equality. Since we also want to provide the reader with some idea of how the interpolation proofs go, in Section 3 we will survey known proofs for several popular logics. In Section 4 and 5 we will prove that our generalized rules are not sound in logic with equality, but they *are sound* in logic without equality. Thus we perhaps also throw some more light on the role of the equality symbol in logic. The question of unrestricted interpolation theorem for classical logic without equality will remain unanswered.

2 Calculi for classical logic, their generalization rules

In Hilbert-style predicate calculi, the generalization rules usually have the following form:

$$\varphi \to \psi \ / \ \exists x\varphi \to \psi \qquad \text{and} \qquad \psi \to \varphi \ / \ \psi \to \forall x\varphi \tag{1}$$

where the variable x has no free occurrences in the formula ψ. An advantage of this variant of the generalization rules is that they do not have to be changed when switching to intuitionistic logic. Hilbert-style calculi also have the instantiation axioms and possibly the equality axioms, both being again the same in classical and in intuitionistic logic. The propositional part of a classical Hilbert-style calculus makes it possible to derive every tautology. Here it is good to recall that tautologies are not the same as logically valid formulas: a predicate formula is a tautology if it can be obtained from a propositional tautology by substituting predicate formulas for atoms. As much about Hilbert-style calculi: in the following we will only need Gentzen-style calculi (that is, sequent calculi).

The rules of a *sequent calculus* derive sequents, not formulas. We prefer the definition where sequent is a pair of finite *sets* (rather than multisets or sequences) of formulas. If a sequent consists of sets Γ and Δ, we write it as $\langle\, \Gamma \;\Rightarrow\; \Delta \,\rangle$ where $\Rightarrow$ is an auxiliary symbol (not a connective) and the angle brackets just separate the sequent from possible other sequents. Its meaning is "if all formulas in Γ hold, then also some formula in Δ holds". The sets Γ and Δ are called *antecedent* and *succedent* of the sequent $\langle\, \Gamma \;\Rightarrow\; \Delta \,\rangle$. A *rule* of a sequent calculus can be binary (if it derives a sequent from a pair of already proved sequents) or unary (if it derives a sequent from one sequent). A *proof* in a sequent calculus is a tree whose nodes are (labeled by) sequents, every leaf (a node having no predecessors) is an initial sequent and every other sequent is derived from its predecessor or from its two predecessors using a rule. A sequent $\langle\, \Gamma \;\Rightarrow\; \Delta \,\rangle$ is *initial* if $\Gamma \cap \Delta \neq \emptyset$ or if $\bot \in \Gamma$. In fact, initial sequents are nulary rules. A proof is a proof of its root, i.e. of its *endsequent*. A proof of a formula φ is a proof of the sequent whose antecedent is empty and whose succedent is $\{\varphi\}$. We write this sequent as $\langle\, \Rightarrow\; \varphi \,\rangle$.

Some rules can be classified as *structural*, i.e. not linked to a logical symbol. The other rules are *logical*. One of the structural rules is *weakening*. It allows adding any formula to antecedent or to succedent. Another structural rule is the *cut rule*, which will be mentioned below. If sequent were defined as a pair of sequences or a pair of multisets, we could also need *contractions* and *exchanges* that make it possible to drop one of two identical formulas or change the order of formulas. Each logical symbol has (logical) rules that "add" a formula in which the symbol occurs at the outermost level. For example, the succedent rules for $\vee$ may look as follows:

$$\frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \vee \psi \,\rangle} \qquad \frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \psi \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \vee \psi \,\rangle} \qquad \frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi, \psi \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \vee \psi \,\rangle.} \tag{2}$$

We follow the usual notation: curly braces enclosing individual formulas are omitted, and commas denote set union, even in expressions like $\mathrm{Symb}(\Gamma, \psi)$. The formula that is "added" by an application of the rule, which in (2) is always $\varphi \vee \psi$, is called *principal formula* of the rule. Once again we have used quotes because the union $\Delta \cup \{\varphi \vee \psi\}$ is legitimate whether $\varphi \vee \psi$ is or is not in Δ, and if it is in Δ, then nothing is added. The formulas that are processed by an application of a rule (the formula φ, the formula ψ and the two formulas φ and ψ in the displayed line (2)) are called *active formulas*. The remaining formulas, which are just copied to the bottom sequent, are *side formulas*.

Given a sequent $\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi, \psi \,\rangle$, one can first apply the first rule in (2) and obtain $\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \vee \psi, \psi \,\rangle$, and then the second rule in (2) yields $\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \vee \psi \,\rangle$.

This reasoning demonstrates that the fact that a principal formula may at the same time be a side formula is very useful, and it also shows that the first two rules, taken together, simulate the third rule. The converse is also true: the third rule can, using a weakening, simulate each of the other two rules.

An example of a binary rule is the antecedent rule for implication. Here we can also opt for one of two variants:

$$\frac{\langle\, \Gamma \Rightarrow \Delta, \varphi \,\rangle \quad \langle\, \Pi, \psi \Rightarrow \Lambda \,\rangle}{\langle\, \Gamma, \Pi, \varphi \to \psi \Rightarrow \Delta, \Lambda \,\rangle} \qquad \frac{\langle\, \Gamma \Rightarrow \Delta, \varphi \,\rangle \quad \langle\, \Gamma, \psi \Rightarrow \Delta \,\rangle}{\langle\, \Gamma, \varphi \to \psi \Rightarrow \Delta \,\rangle.} \tag{3}$$

In both cases a sequent containing an implication $\varphi \to \psi$ is derived from two sequents, one containing φ in the succedent and another containing ψ in the antecedent. The difference is that in the second rule in (3) the two upper sequents have the same sets (the sets Γ and Δ) of side formulas. It is the *context-sensitive* variant of the rule, while the first rule, having four sets Γ, Δ, Π and Λ of side formulas, is *context-insensitive*. It is clear that the two variants are equivalent (mutually simulable): the context-insensitive variant admits the case where $\Gamma = \Pi$ and $\Delta = \Lambda$, and the context-sensitive variant can simulate the context-insensitive variant with the help of some weakenings.

We do not list the remaining propositional logical rules: the succedent rule for implication, the antecedent rule for disjunction (here one can again opt for a context-sensitive or context-insensitive variant) and the rules for conjunction. The reader may guess (design) them, or they can be found in the literature. Worth mentioning is the *cut rule*:

$$\frac{\langle\, \Gamma \Rightarrow \Delta, \varphi \,\rangle \quad \langle\, \Pi, \varphi \Rightarrow \Lambda \,\rangle}{\langle\, \Gamma, \Pi \Rightarrow \Delta, \Lambda \,\rangle,} \tag{4}$$

which makes it possible to drop a formula if it occurs in the succedent of a proved sequent and in the antecedent of another already proved sequent. A proof not containing an application of the cut rule is a *cut-free proof*. Inspection of the rules other than the cut rule shows that every formula in a cut-free proof is a subformula (in predicate logic, a substitution instance of a subformula) of some formula in the endsequent. Cut-free proofs formalize "direct reasoning", not containing detours through unrelated formulas. Classical logic, both propositional and predicate, satisfies the *cut-elimination theorem*: every provable sequent is provable without using the cut rule. The questions whether the cut-elimination theorem holds, or whether a sequent calculus exists at all, is relevant and studied for every logic.

As to classical logic, we use GK to denote its (more or less just described) calculus. The letters stand for "Gentzen klassisch". In the literature one can also find LK where L refers to "logic". We use the same name GK also for the *predicate* version of the classical calculus, which we will deal with now. The generalization rules of GK are

$$\frac{\langle\, \Gamma, \varphi_x(y) \Rightarrow \Delta \,\rangle}{\langle\, \Gamma, \exists x \varphi \Rightarrow \Delta \,\rangle} \quad \text{and} \quad \frac{\langle\, \Gamma \Rightarrow \Delta, \varphi_x(y) \,\rangle}{\langle\, \Gamma \Rightarrow \Delta, \forall x \varphi \,\rangle} \tag{5}$$

where $\varphi_x(y)$ denotes the result of substituting y for all free occurrences of the variable x in φ, and y, the *eigenvariable*, is a variable substitutable for x in φ that has no free occurrences in the resulting sequent $\langle\, \Gamma, \exists x \varphi \Rightarrow \Delta \,\rangle$ or $\langle\, \Gamma \Rightarrow \Delta, \forall x \varphi \,\rangle$. Thus Hilbert-style calculi and sequent calculi share a restriction concerning the variable that is generalized. The rules (5), furthermore, make it possible to rename this variable. This difference is not essential: while the rules (1) do not allow renaming, in a Hilbert-style calculus renaming of bound variables can, of course, be achieved. The remaining quantifier rules of GK are the instantiation (or specification) rules:

$$\frac{\langle\, \Gamma, \varphi_x(t) \;\Rightarrow\; \Delta \,\rangle}{\langle\, \Gamma, \forall x \varphi \;\Rightarrow\; \Delta \,\rangle} \qquad \text{and} \qquad \frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi_x(t) \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \exists x \varphi \,\rangle} \tag{6}$$

where, again, $\varphi_x(t)$ denotes the result of substituting t for all free occurrences of x in φ, and t is a term of the language in question that is substitutable for x in φ. It is good to notice the common properties and the differences between the generalization and instantiation rules. In both (5) and (6) a quantified formula is obtained by "unsubstituting" a substitutable term. However, in (5) this term must be a variable and it must not occur in the resulting sequent. The latter stipulation is called *eigenvariable condition*, and it is easy to verify that without it the rules (5) would not be sound with respect to the classical (i.e. Tarskian) semantics.

The generalization rules correspond to reasoning that appears in virtually every mathematical proof. For example, the second rule in (5) formalizes the following argument.

> We have to show that every individual has the property φ. Let an individual y be given. [...]. Therefore, y has the property φ. Since y was arbitrary, all individuals have the property φ.

This reasoning is sound if y is a *new* variable, i.e. if y does not denote anything else in the proof in question. And this is exactly the stipulation to which the eigenvariable condition corresponds. The first rule in (5) corresponds to a logical step that frequently occurs as well. This is not a surprise since in classical logic the quantifiers $\exists$ and $\forall$ behave symmetrically and are interdefinable.

In this paper we consider the following *enhanced*, or generalized, *generalization rules*:

$$\frac{\langle\, \Gamma, \varphi_{x_1,..,x_n}(t_1,..,t_n) \;\Rightarrow\; \Delta \,\rangle}{\langle\, \Gamma, \exists x_1 \ldots \exists x_n \varphi \;\Rightarrow\; \Delta \,\rangle} \qquad \text{and} \qquad \frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi_{x_1,..,x_n}(t_1,..,t_n) \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \forall x_1 \ldots \forall x_n \varphi \,\rangle} \tag{7}$$

where $t_1, .., t_n$ are pairwise different terms that are substitutable for $x_1, .., x_n$ in φ and such that, for each i, the outermost function symbol of t_i (the term t_i itself if it is a variable) has no occurrences (has no free occurrences) in the resulting sequent (in the bottom). The terms t_i can contain inner occurrences of arbitrary function symbols and of arbitrary variables. We cannot claim that these enhanced rules correspond to some logical steps in real proofs. Indeed, we never write something so strange like this:

> We have to show that every individual x is in the relation φ to z, i.e. that it satisfies $\varphi(x, z)$. Let an individual be given and let us denote it by $G(z)$. [...]. Since $G(z)$ is in the relation φ to z, we indeed have $\forall x \varphi(x, z)$.

However, modified generalization rules like (7) can be useful when thinking about interpolation in predicate logic and about its proof-theoretic proofs.

3 Aspects of interpolation proofs

In the literature there exist both semantic and proof-theoretic proofs of interpolation theorems. In this section we will survey known proof-theoretic proofs for several popular logics. We will also reproduce a proof (known from various sources like [Tak75] and [Bus98]) for classical predicate logic without function symbols of nonzero arity.

A proof-theoretic proof of an interpolation theorem usually consists in two steps: first finding a sequent form of the theorem, i.e. formulating a claim concerning provable sequents, and then proving that claim by induction on the depth of a cut-free proof $\mathcal{P}$. The steps presuppose that the completeness theorem and the cut-elimination theorem hold for the given logic. In the case of *classical propositional logic*, where we know that a formula φ is a tautology if and only if the sequent $\langle \Rightarrow \varphi \rangle$ is provable in GK and that the cut-elimination theorem is true for GK, the claim can be as follows. *Let $\mathcal{P}$ be a cut-free proof of $\langle \Gamma; \Pi \Rightarrow \Delta; \Lambda \rangle$. Then there exists a formula μ such that* $\mathrm{Symb}(\mu) \subseteq (\Gamma \cup \Delta) \cap (\Pi \cup \Lambda)$ *and both* $\langle \Gamma \Rightarrow \Delta, \mu \rangle$ *and* $\langle \Pi, \mu \Rightarrow \Lambda \rangle$ *are provable*. The semicolons denote set union just like commas, but in addition they indicate how the given sequent is divided into two sequents $\langle \Gamma \Rightarrow \Delta \rangle$ and $\langle \Pi \Rightarrow \Lambda \rangle$. The sets Γ and Π and also the sets Δ and Λ do not have to be disjoint. Once this claim is proved, the interpolation theorem follows: if $\varphi \to \psi$ is a tautology, then $\langle \varphi; \Rightarrow ; \psi \rangle$ is provable, and then a formula μ obtained by the claim is an interpolant of φ and ψ.

If an initial sequent, i.e. an endsequent of a zero-depth proof, is divided into two sequents, we have one of the following six situations. Recall the agreement that $\bot$ is a basic symbol and that $\top$ and $\neg\varphi$ are considered shorthands for $\bot \to \bot$ and $\varphi \to \bot$:

$$\begin{array}{ll} \langle \Gamma, \bot; \Pi \Rightarrow \Delta; \Lambda \rangle, & \langle \Gamma; \bot, \Pi \Rightarrow \Delta; \Lambda \rangle \\ \langle \Gamma, \varphi; \Pi \Rightarrow \Delta; \varphi, \Lambda \rangle, & \langle \Gamma; \varphi, \Pi \Rightarrow \Delta; \varphi, \Lambda \rangle \\ \langle \Gamma, \varphi; \Pi \Rightarrow \Delta, \varphi; \Lambda \rangle, & \langle \Gamma; \varphi, \Pi \Rightarrow \Delta, \varphi; \Lambda \rangle. \end{array} \tag{8}$$

One can easily check that the six formulas $\bot$, $\bot \to \bot$, φ, $\bot \to \bot$, $\bot$ and $\varphi \to \bot$, respectively, satisfy the requirements on interpolant. For example in the third case every extralogical symbol in φ occurs in both $\mathrm{Symb}(\Gamma, \varphi, \Delta)$ and $\mathrm{Symb}(\Pi, \varphi, \Lambda)$ and both $\langle \Gamma, \varphi \Rightarrow \Delta, \varphi \rangle$ where φ is added to the succedent, and $\langle \Pi, \varphi \Rightarrow \varphi, \Lambda \rangle$ where φ is added to the antecedent, are provable. In the first case both $\langle \Gamma, \bot \Rightarrow \Delta, \bot \rangle$ and $\langle \Pi, \bot \Rightarrow \Lambda \rangle$ are provable, and the stipulation concerning symbols is satisfied because $\mathrm{Symb}(\bot) = \emptyset$. Notice also that the last case would be problematic in intuitionistic logic. The provability of $\langle \Gamma \Rightarrow \Delta, \varphi, \varphi \to \bot \rangle$ is based on the provability of $\langle \Rightarrow \varphi, \varphi \to \bot \rangle$, and the latter sequent is in fact the same as the disjunction $\varphi \vee \neg\varphi$.

We proceed to the induction step. Let a nonzero-depth cut-free proof $\mathcal{P}$ of a sequent divided by semicolons into two subsequents be given. Distinguish the cases whether the last inference in $\mathcal{P}$ is an application of one or another rule and whether the principal formula of that inference is before or after a semicolon. For example, if the last inference of $\mathcal{P}$ is the antecedent $\to$-rule and its principal formula $\varphi \to \psi$ is after the semicolon, we have:

$$\frac{\langle \Gamma; \Pi \Rightarrow \Delta; \varphi, \Lambda \rangle \quad \langle \Gamma; \psi, \Pi \Rightarrow \Delta; \Lambda \rangle}{\langle \Gamma; \varphi \to \psi, \Pi \Rightarrow \Delta; \Lambda \rangle.} \tag{9}$$

We for simplicity assume that the binary rules of our calculus are context-sensitive. The depths of the subproofs $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\langle\, \Gamma; \Pi \;\Rightarrow\; \Delta; \varphi, \Lambda \,\rangle$ and $\langle\, \Gamma; \psi, \Pi \;\Rightarrow\; \Delta; \Lambda \,\rangle$ are less than the depth of $\mathcal{P}$ and thus the induction hypothesis is applicable. It says that if we arbitrarily divide the endsequents of $\mathcal{P}_1$ and $\mathcal{P}_2$ into subsequents, then a required formula exists. We do not have to be creative when dividing the two endsequents: since it is given that $\varphi\!\to\!\psi$ is after the semicolon, in the upper sequents we just put the semicolons before the active formulas φ and ψ. Let ε and ν be interpolants of the endsequents of $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Thus the following four sequents are provable:

$$\begin{array}{ll} \langle\, \Gamma \;\Rightarrow\; \Delta, \varepsilon \,\rangle, & \langle\, \Pi, \varepsilon \;\Rightarrow\; \varphi, \Lambda \,\rangle, \\ \langle\, \Gamma \;\Rightarrow\; \Delta, \nu \,\rangle, & \langle\, \psi, \Pi, \nu \;\Rightarrow\; \Lambda \,\rangle. \end{array} \tag{10}$$

We have not written down the succedent &-rule, but it is natural and makes it possible to derive $\langle\, \Gamma \;\Rightarrow\; \Delta, \varepsilon \,\&\, \nu \,\rangle$ from the first and third sequents. Using the antecedent &-rule, the sequents $\langle\, \Pi, \varepsilon \,\&\, \nu \;\Rightarrow\; \varphi, \Lambda \,\rangle$ and $\langle\, \psi, \Pi, \varepsilon \,\&\, \nu \;\Rightarrow\; \Lambda \,\rangle$ can be obtained from the second and fourth sequent respectively, and they yield $\langle\, \Pi, \varphi \to \psi, \varepsilon \,\&\, \nu \;\Rightarrow\; \Lambda \,\rangle$ using the antecedent implication rule. Since every atom in ε is in both $\mathrm{Symb}(\Gamma, \Delta)$ and $\mathrm{Symb}(\Pi, \varphi, \Lambda)$, and every atom in ν is in both $\mathrm{Symb}(\Gamma, \Delta)$ and $\mathrm{Symb}(\psi, \Pi, \Lambda)$, it is clear that the formula $\varepsilon \,\&\, \nu$ is built up only from atoms that occur in both $\mathrm{Symb}(\Gamma, \Delta)$ and $\mathrm{Symb}(\Pi, \varphi \to \psi, \Lambda)$. We see that the conjunction $\varepsilon \,\&\, \nu$ satisfies all requirements, and thus it is an interpolant of $\langle\, \Gamma; \varphi \to \psi, \Pi \;\Rightarrow\; \Delta; \Lambda \,\rangle$.

All other cases are treated similarly. In the case of a binary rule, the conjunction or the disjunction of the interpolants of the upper sequents always works as an interpolant of the endsequent of the whole proof $\mathcal{P}$. In the case of a unary rule an interpolant of the upper sequent satisfies the requirements for an interpolant of the endsequent.

In the definition of the calculus GK one can insist that the principal formulas of initial sequents be atomic. From this fact one can obtain a somewhat stronger version of the interpolation theorem for classical propositional logic: for any two formulas φ and ψ such that $\varphi\!\to\!\psi$ is a tautology there exists an interpolant built up from atoms and negated atoms using conjunctions and disjunctions only.

In *modal logic* we have an additional unary logical symbol $\Box$. A formula $\Box\varphi$ is read "necessarily φ". Besides $\Box$, the necessity operator, one can also consider $\Diamond$, the possibility operator. However, it is usually considered a defined symbol: $\Diamond\varphi$ is a shorthand for $\neg\Box\neg\varphi$. One of extensively studied propositional modal logics is *provability logic*. Different symbolic names for this logic can be found in the literature. Now, after about fifty years history, it is usually denoted by GL where the letters refer to Gödel and Löb. The semantics (one of semantics) for GL is based on the idea to understand the $\Box$ operator (interpret it, translate it to) *provability* in some recursively axiomatized and sufficiently strong axiomatic theory, *formalized* in the same (or sometimes different) axiomatic theory. In GL one can model reasoning about self-referential sentences, and GL also has some applications in this field and thus in meta-mathematics. One of these applications is that, under some circumstances, a sentences defined by self-reference is unique up to provable equivalence.

Hilbert-style calculus for provability logic is based on the axioms K and 4 that traditionally occur in say more philosophically oriented literature, and on the Löb's axiom schema $\Box(\Box\varphi \to \varphi) \to \Box\varphi$. Sequent calculus for GL was defined in [SV82]. It is based on a single modal rule, which is sufficient to simulate the axioms K and 4 as well as the Löb's axiom:

$$\frac{\langle\, \Gamma, \Box\Gamma, \Box\varphi \;\Rightarrow\; \varphi \,\rangle}{\langle\, \Box\Gamma \;\Rightarrow\; \Box\varphi \,\rangle.} \tag{11}$$

Here $\Box\Gamma$ denotes the set $\{\, \Box\psi \;;\; \psi \in \Gamma \,\}$. The rule is applicable on a sequent $\mathcal{S}$ only if (i) the succedent of $\mathcal{S}$ consists of *exactly one* formula φ, (ii) the antecedent of $\mathcal{S}$ contains $\Box\varphi$, and (iii) the rest of the antecedent consists of pairs ψ and $\Box\psi$. The conditions (ii) and (iii) are not really demanding because one can always add some formulas using the weakening rule. All formulas in the bottom sequent of (11) begin with $\Box$.

The sequent calculus for GL satisfies cut-elimination, and the interpolation theorem for GL can be proved along the same lines as for classical propositional logic. That is, we prove the same claim concerning a sequent $\langle\, \Gamma; \Pi \;\Rightarrow\; \Delta, \Lambda \,\rangle$ by induction on the depth of its cut-free proof $\mathcal{P}$. Most cases are the same as above, but there are two additional cases to consider: if the last inference of $\mathcal{P}$ is an application of the modal rule and its principal formula $\Box\varphi$ occurs before, or after the semicolon. Let us discuss the former case, the latter is treated similarly. The endsequent of $\mathcal{P}$ thus has the form $\langle\, \Box\Gamma; \Box\Pi \;\Rightarrow\; \Box\varphi; \,\rangle$. The sequent to which the modal rule is applied must have Γ, $\Box\Gamma$, Π, $\Box\Pi$ and $\Box\varphi$ in the antecedent and φ in the succedent, and we apply the induction hypothesis (in the expected way) on the sequent $\langle\, \Gamma, \Box\Gamma, \Box\varphi; \Pi, \Box\Pi \;\Rightarrow\; \varphi; \,\rangle$. Thus there exists a formula ν and proofs $\mathcal{P}_1$ and $\mathcal{P}_2$ of the sequents $\langle\, \Gamma, \Box\Gamma, \Box\varphi \;\Rightarrow\; \varphi, \nu \,\rangle$ and $\langle\, \Pi, \Box\Pi, \nu \;\Rightarrow\; \,\rangle$. The proofs $\mathcal{P}_1$ and $\mathcal{P}_2$ can be extended as follows:

$$\cfrac{\cfrac{\cfrac{\langle\, \Gamma, \Box\Gamma, \Box\varphi \;\Rightarrow\; \varphi, \nu \,\rangle}{\langle\, \Gamma, \Box\Gamma, \Box\varphi, \neg\nu, \Box\neg\nu \;\Rightarrow\; \varphi \,\rangle}}{\langle\, \Box\Gamma, \Box\neg\nu \;\Rightarrow\; \Box\varphi \,\rangle}}{\langle\, \Box\Gamma \;\Rightarrow\; \Box\varphi, \neg\Box\neg\nu \,\rangle} \qquad \cfrac{\cfrac{\cfrac{\langle\, \Pi, \Box\Pi, \nu \;\Rightarrow\; \,\rangle}{\langle\, \Pi, \Box\Pi, \Box\neg\nu \;\Rightarrow\; \neg\nu \,\rangle}}{\langle\, \Box\Pi \;\Rightarrow\; \Box\neg\nu \,\rangle}}{\langle\, \Box\Pi, \neg\Box\neg\nu \;\Rightarrow\; \,\rangle}$$

In the left we have first negated ν and moved it to the other side of the sequent. This is exactly what the $\neg$-rules do. We have also added the formula $\Box\neg\nu$ via the weakening rule, and we did it in one line to save space. Then the modal rule is applicable, and finally the endsequent is obtained by another application of the $\neg$-rule. The explanation for the proof in the right is similar. Since every symbol (every atom) in ν is in both $\mathrm{Symb}(\Gamma, \Box\Gamma, \Box\varphi, \varphi)$ and $\mathrm{Symb}(\Pi, \Box\Pi)$, it is clear that every atom in $\neg\Box\neg\nu$ is in both $\mathrm{Symb}(\Box\Gamma, \Box\varphi)$ and $\mathrm{Symb}(\Box\Pi)$. Thus $\mu = \neg\Box\neg\nu$ is as required.

The above formal proofs can be easily modified for the case where $\neg$ is not considered a basic symbol. However, the presence of $\bot$ is essential in GL. Without it, the formulas $\varphi = \Box(p \mathbin{\&} \neg p)$ and $\psi = \Box(q \mathbin{\&} \neg q)$, of which φ is not refutable and ψ is not provable in GL, would have no interpolant.

A possible exercise could be this: take $\varphi = \neg\Box p$ and $\psi = \Box(q \to \neg\Box q) \to \neg\Box q$, prove $\varphi \to \psi$ in the sequent calculus and find an interpolant of these two formulas. The choice of ψ is motivated by Gödel's first incompleteness theorem: if a sentence q is provably equivalent to its own unprovability, or, if it just *implies* its own uprovability, then it is unprovable.

Provability logic has a satisfactory Kripke semantics. Its Hilbert-style and sequent calculi polynomially simulate each other and are complete with respect to transitive reversely well-founded trees, and also with respect to (the smaller class of all) finite transitive and irreflexive trees. To show completeness of the sequent calculus with respect to Kripke semantics, one can prove (Sambin and Valentini in [SV82] prove) the following claim: every sequent either has a Kripke counterexample, or a cut-free proof. This way the completeness and the cut-elimination theorem are proved at the same time. Similarly, i.e. via a semantic detour, one can actually prove the cut-elimination theorem for each logic mentioned in this paper. A direct proof for GL, i.e. an algorithm that, given a proof, outputs a cut-free proof of the same sequent, was published in [GR08]. GL is also complete with respect to the arithmetic semantics. This is a famous Solovay's result published in [Sol76].

A sequent calculus for *intuitionistic logic* can be obtained by the following modification of GK: the succedent rules for $\neg$, $\rightarrow$ and $\forall$ do not admit side formulas in succedent. Thus after one of these rules is used, the succedent is a singleton consisting of the principal formula. We call this calculus GJ, where G again refers to Gentzen. Many authors (like Takeuti in [Tak75]) use LJ to denote this calculus. A related calculus GJ^1 is based on an even stronger restriction: each succedent in a GJ^1-proof must be empty or a singleton. Our assumption that $\neg$ is a defined symbol again simplifies matters, and it also has the following consequence: each succedent in a GJ^1-proof contains *exactly one* formula. This is so because no rule except the $\neg$-rules can change the number of formulas in succedent. Thus the $\rightarrow$-rules of GJ^1 are

$$\frac{\langle\, \Gamma \Rightarrow \varphi \,\rangle \quad \langle\, \Gamma, \psi \Rightarrow \delta \,\rangle}{\langle\, \Gamma, \varphi \rightarrow \psi \Rightarrow \delta \,\rangle} \qquad\qquad \frac{\langle\, \Gamma, \varphi \Rightarrow \psi \,\rangle}{\langle\, \Gamma \Rightarrow \varphi \rightarrow \psi \,\rangle}$$

where the rule in the right, the succedent implication rule, is the same as in GJ. The &-rules of GJ^1 are:

$$\frac{\langle\, \Gamma, \varphi \Rightarrow \delta \,\rangle}{\langle\, \Gamma, \varphi \,\&\, \psi \Rightarrow \delta \,\rangle} \qquad \frac{\langle\, \Gamma, \psi \Rightarrow \delta \,\rangle}{\langle\, \Gamma, \varphi \,\&\, \psi \Rightarrow \delta \,\rangle} \qquad \frac{\langle\, \Gamma \Rightarrow \varphi \,\rangle \quad \langle\, \Gamma \Rightarrow \psi \,\rangle}{\langle\, \Gamma \Rightarrow \varphi \,\&\, \psi \,\rangle.}$$

The completeness and cut-elimination theorems hold for both GJ and GJ^1. It is not clear (to the present author) how about GJ, but GJ^1 can be used to prove the interpolation theorem for intuitionistic propositional logic. We follow the proof in [Min02].

As in other cases, we proceed by induction on the depth of a cut-free proof. However, the claim we prove is now different: *for any cut-free proof of a sequent* $\langle\, \Gamma; \Pi \Rightarrow \lambda \,\rangle$ *in the calculus* GJ^1 *there exists a formula* μ *such that* $\langle\, \Gamma \Rightarrow \mu \,\rangle$ *and* $\langle\, \Pi, \mu \Rightarrow \lambda \,\rangle$ *are provable and all atoms in* μ *are in both* $\mathrm{Symb}(\Gamma)$ *and* $\mathrm{Symb}(\Pi, \lambda)$. Now there are no semicolons in succedents. To prove this claim is sufficient for our goal: to find an interpolant of a pair φ and ψ, it is enough to put $\Gamma = \{\varphi\}$, $\Pi = \emptyset$ and $\lambda = \psi$. The base case is as follows. If $\langle\, \Gamma; \Pi \Rightarrow \lambda \,\rangle$ is an initial sequent (the endsequent of a zero-depth proof $\mathcal{P}$), we deal with the following four cases:

$$\langle\, \Gamma, \bot; \Pi \Rightarrow \lambda \,\rangle, \quad \langle\, \Gamma; \bot, \Pi \Rightarrow \lambda \,\rangle, \quad \langle\, \Gamma, \varphi; \Pi \Rightarrow \lambda \,\rangle, \quad \langle\, \Gamma; \varphi, \Pi \Rightarrow \lambda \,\rangle$$

and it is straightforward to verify that $\bot$, $\bot \rightarrow \bot$, φ and $\bot \rightarrow \bot$ can be picked for μ.

As to the induction step, most cases are the same as in classical logic. For example, if the last inference in a proof $\mathcal{P}$ derives $\langle\, \Gamma; \varphi \to \psi, \Pi \;\Rightarrow\; \lambda \,\rangle$ from a sequent whose succedent is $\{\varphi\}$ and from another sequent having ψ in the antecedent, we write the two sequents as $\langle\, \Gamma; \Pi \;\Rightarrow\; \varphi \,\rangle$ and $\langle\, \Gamma; \psi, \Pi \;\Rightarrow\; \lambda \,\rangle$. Then the induction step yields formulas ε and ν such that all atoms in ε are in both $\mathrm{Symb}(\Gamma)$ and $\mathrm{Symb}(\Pi, \varphi)$ all atoms in ν are in both $\mathrm{Symb}(\Gamma)$ and $\mathrm{Symb}(\psi, \Pi, \lambda)$ and the sequents $\langle\, \Gamma \;\Rightarrow\; \varepsilon \,\rangle$, $\langle\, \Pi, \varepsilon \;\Rightarrow\; \varphi \,\rangle$, $\langle\, \Gamma \;\Rightarrow\; \nu \,\rangle$ and $\langle\, \psi, \Pi, \nu \;\Rightarrow\; \lambda \,\rangle$ are provable. Then it is easy to verify that $\mu = \varepsilon \,\&\, \nu$ is as required.

A case that cannot be simply copied from classical logic is when the endsequent of $\mathcal{P}$ is given as $\langle\, \Gamma, \varphi \to \psi; \Pi \;\Rightarrow\; \lambda \,\rangle$ with $\varphi \to \psi$ being a principal formula. In this case we use the right to divide the two preceding sequents as needed, and we write them as $\langle\, \Pi; \Gamma \;\Rightarrow\; \varphi \,\rangle$ and $\langle\, \Gamma, \psi; \Pi \;\Rightarrow\; \lambda \,\rangle$. The induction hypothesis yields formulas ε and ν such that the sequents

$$\langle\, \Pi \;\Rightarrow\; \varepsilon \,\rangle, \quad \langle\, \Gamma, \varepsilon \;\Rightarrow\; \varphi \,\rangle, \quad \langle\, \Gamma, \psi \;\Rightarrow\; \nu \,\rangle, \quad \langle\, \Pi, \nu \;\Rightarrow\; \lambda \,\rangle$$

are provable. Then from the second and third, and from the first and fourth of them we can continue as follows:

$$\dfrac{\dfrac{\langle\, \Gamma, \varepsilon \;\Rightarrow\; \varphi \,\rangle \qquad \langle\, \Gamma, \psi \;\Rightarrow\; \nu \,\rangle}{\langle\, \Gamma, \varphi \to \psi, \varepsilon \;\Rightarrow\; \nu \,\rangle}}{\langle\, \Gamma, \varphi \to \psi \;\Rightarrow\; \varepsilon \to \nu \,\rangle} \qquad\qquad \dfrac{\langle\, \Pi \;\Rightarrow\; \varepsilon \,\rangle \qquad \langle\, \Pi, \nu \;\Rightarrow\; \lambda \,\rangle}{\langle\, \Pi, \varepsilon \to \nu \;\Rightarrow\; \lambda \,\rangle}$$

Since all atoms in ε are in both $\mathrm{Symb}(\Pi)$ and $\mathrm{Symb}(\Gamma, \varphi)$ and all atoms in ν are in both $\mathrm{Symb}(\Gamma, \psi)$ and $\mathrm{Symb}(\Pi, \lambda)$, we see that all atoms in $\varepsilon \to \nu$ are in both $\mathrm{Symb}(\Gamma, \varphi \to \psi)$ and $\mathrm{Symb}(\Pi, \lambda)$. Thus $\mu = \varepsilon \to \nu$ is as required.

In *classical predicate logic without equality* we can stick with the same claim as in classical propositional logic, but we have to consider the generalization rules (5) and the instantiation rules (6). Generalization poses no problem. Indeed, let a proof $\mathcal{P}$ of a sequent divided by semicolons be given, let its last inference be an application of the antecedent $\exists$-rule with a principal formula $\exists x \varphi$:

$$\frac{\langle\, \Gamma; \varphi_x(y), \Delta \;\Rightarrow\; \Pi; \Lambda \,\rangle}{\langle\, \Gamma; \exists x \varphi, \Delta \;\Rightarrow\; \Pi; \Lambda \,\rangle} \tag{12}$$

and let ν be such that $\langle\, \Gamma \;\Rightarrow\; \Pi, \nu \,\rangle$ and $\langle\, \varphi_x(y), \Delta, \nu \;\Rightarrow\; \Lambda \,\rangle$ are provable and $\mathrm{Symb}(\nu)$ is a subset of both $\mathrm{Symb}(\Gamma, \Pi)$ and $\mathrm{Symb}(\varphi_x(y), \Delta, \Lambda)$. From the fact that the variable y satisfies the eigenvariable condition we can draw several consequences. (i) Since y is not free in formulas in Γ and Π, from $\mathrm{Symb}(\nu) \subseteq \mathrm{Symb}(\Gamma, \Pi)$ it is clear that y is not free in ν. (ii) Once we know that, from $\mathrm{Symb}(\nu) \subseteq \mathrm{Symb}(\varphi_x(y), \Delta, \Lambda)$ we obtain $\mathrm{Symb}(\nu) \subseteq \mathrm{Symb}(\exists x \varphi, \Delta, \Lambda)$. And (iii), since y is not free in the endsequent of (12), the following is a valid inference according to the antecedent $\exists$-rule:

$$\frac{\langle\, \varphi_x(y), \Delta, \nu \;\Rightarrow\; \Lambda \,\rangle}{\langle\, \exists x \varphi, \Delta, \nu \;\Rightarrow\; \Lambda \,\rangle.}$$

Thus the formula ν, without any modification, satisfies the requirements. Reasoning in the other cases (principal formula in front of a semicolon or the succedent $\forall$-rule as the last step in $\mathcal{P}$) is completely analogous.

Assume now that the last step in $\mathcal{P}$ is an application of one of the rules (6). In addition, assume that the term t is a variable, say y. We thus have a situation like this:

$$\frac{\langle\,\Gamma;\varphi_x(y),\Pi \;\Rightarrow\; \Delta;\Lambda\,\rangle}{\langle\,\Gamma;\forall x\varphi,\Pi \;\Rightarrow\; \Delta;\Lambda\,\rangle.} \tag{13}$$

It looks similar to (12), but now y may occur free in any formula in the endsequent. Let again ν be a formula guaranteed by the induction hypothesis. If y is free in $\exists x\varphi$ or in a formula in Π or Λ, then $\mathrm{Symb}(\forall x\varphi,\Pi,\Lambda) = \mathrm{Symb}(\varphi_x(y),\Pi,\Lambda)$ and, no matter whether y is free in it, the formula ν can be taken as the formula required for the endsequent. Otherwise we have $\mathrm{Symb}(\forall x\varphi,\Pi,\Lambda) = \mathrm{Symb}(\varphi_x(y),\Pi,\Lambda) - \{y\}$. Then

$$\frac{\langle\,\Gamma \;\Rightarrow\; \Delta,\nu\,\rangle}{\langle\,\Gamma \;\Rightarrow\; \Delta,\exists y\nu\,\rangle} \qquad \text{and} \qquad \cfrac{\cfrac{\langle\,\varphi_x(y),\Pi,\nu \;\Rightarrow\; \Lambda\,\rangle}{\langle\,\forall x\varphi,\Pi,\nu \;\Rightarrow\; \Lambda\,\rangle}}{\langle\,\forall x\varphi,\Pi,\exists y\nu \;\Rightarrow\; \Lambda\,\rangle}$$

are valid inferences because, in the second step in the right, the eigenvariable condition for y is met. Thus $\mu = \exists y\nu$ is a formula required for the endsequent of (13).

The problematic case is when we have a complex term t in the place of the variable y in (13). Then t may contain several symbols (function symbols and variables) that are not in $\mathrm{Symb}(\forall x\varphi,\Pi,\Lambda)$. These may occur in the formula ν, but must not occur in the formula needed for the endsequent of (13). This case, while unresolved, is the main reason for writing this paper.

4 The presence or absence of the equality symbol

Let L be the language $\{P,R,G\}$ where P is a unary predicate, R is a binary predicate and G is a unary function symbol, and consider the sequent

$$\begin{aligned}\langle\, R(G(G(z)),G(z)), \forall x\forall y(P(x)\ \&\ P(y) \to x = y)^{(\alpha)},\\ \forall x\forall y(R(x,y) \to \neg P(x)\ \&\ P(y))^{(\beta)} \;\Rightarrow\; \neg P(z)\,\rangle\end{aligned} \tag{14}$$

It is easy to verify that in logic with equality this sequent is logically valid:

> From $R(G(G(z)),G(z))$ we obtain $\neg P(G(G(z)))$ and $P(G(z))$ using the sentence β in the antecedent of (14). Assume that $P(z)$. Then $P(G(z))$ together with α yield $z = G(z)$. From this we obtain $G(z) = G(G(z))$, and then from $P(G(z))$ we have $P(G(G(z)))$, which is is a contradiction.

Now consider the sequent

$$\begin{aligned}\langle\, \exists u\exists v R(u,v), \forall x\forall y(P(x)\ \&\ P(y) \to x = y),\\ \forall x\forall y(R(x,y) \to \neg P(x)\ \&\ P(y)) \;\Rightarrow\; \neg P(z)\,\rangle\end{aligned} \tag{15}$$

Since G does not occur in it, it can be derived from (14) using the left rule in (7). However, it is straightforward to see that it is not logically valid. For this it is sufficient to pick a two-element structure $\mathcal{D}$ with a domain $D = \{a,b\}$ such that $R^{\mathcal{D}} = \{[a,b]\}$

and $P^{\mathcal{D}} = \{b\}$, and evaluate the variable z by b. This example shows that the rules (7) are not sound with respect to the classical semantics for logic with equality. In the following theorem and in its proof we write $\underline{x}$ and $\underline{t}$ to denote an n-tuple.

Theorem 1 *Let φ be a formula, let $x_1, .., x_n$ be distinct variables and let $t_1, .., t_n$ be distinct terms such that every t_i is substitutable for x_i in φ. Furthermore, assume that if t_i is a variable, then it has no free occurrences in $\Gamma \cup \Delta \cup \{\exists \underline{x} \varphi\}$, and if t_i is a complex term, then its outermost symbol does not occur in $\Gamma \cup \Delta \cup \{\exists \underline{x} \varphi\}$. Then, in logic without equality, if $\langle\, \Gamma, \varphi_{\underline{x}}(\underline{t}) \;\Rightarrow\; \Delta \,\rangle$ is logically valid, then $\langle\, \Gamma, \exists \underline{x} \varphi \;\Rightarrow\; \Delta \,\rangle$ is logically valid, and if $\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi_{\underline{x}}(\underline{t}) \,\rangle$ is logically valid, then $\langle\, \Gamma \;\Rightarrow\; \Delta, \forall \underline{x} \varphi \,\rangle$ is logically valid.*

Proof Since the two claims are symmetric, it is sufficient to deal with the existential quantification. Let L be the set of all function and predicate symbols in $\langle\, \Gamma, \exists \underline{x} \varphi \;\Rightarrow\; \Delta \,\rangle$. Let $G_1, .., G_m$ be the (distinct) function symbols that appear in $t_1, .., t_n$ as the outermost symbols, and let $y_1, .., y_k$ be those t_i that are variables. The symbols G_j are not in L. The terms $t_1, .., t_n$ may contain inner occurrences of further function symbols (the symbols G_j included) and of variables (the variables y_j included). Some of them may share the outermost symbol. We assume that $\langle\, \Gamma, \exists \underline{x} \varphi \;\Rightarrow\; \Delta \,\rangle$ is not logically valid and we aim to show that $\langle\, \Gamma, \varphi_{\underline{x}}(\underline{t}) \;\Rightarrow\; \Delta \,\rangle$ is not logically valid either. We thus start with a semantic counterexample for $\langle\, \Gamma, \exists \underline{x} \varphi \;\Rightarrow\; \Delta \,\rangle$. It consists of a structure $\mathcal{D}$ for L and a valuation e_0 of variables in $\mathcal{D}$ such that $\mathcal{D} \models \psi[e_0]$ for every $\psi \in \Gamma \cup \{\exists \underline{x} \varphi\}$ and $\mathcal{D} \not\models \psi[e_0]$ for every $\psi \in \Delta$. Note that we use square brackets to enclose a valuation when writing the relation "satisfies" symbolically. Since $\mathcal{D} \models (\exists \underline{x} \varphi)[e_0]$, we can fix elements $a_1, .., a_n$ of the domain D of $\mathcal{D}$ such that

$$\mathcal{D} \models \varphi[e_0(x_1/a_1, .., x_n/a_n)]. \tag{i}$$

Here $e_0(x_1/a_1, .., x_n/a_n)$ denotes the valuation that maps $x_1, .., x_n$ to $a_1, .., a_n$ and agrees with e_0 at all other variables. Let U be the set of all terms in $L \cup \{G_1, .., G_m\}$. We put $M = D \times U$ and we fix an arbitrary $a_0 \in D$. The realization $F^{\mathcal{M}}$ of an r-ary function symbol $F \in L$, the realization $R^{\mathcal{M}}$ of an r-ary relation symbol $R \in L$, and the realizations $G_j^{\mathcal{M}}$ of the symbols $G_1, .., G_m$ are defined as follows:

$$F^{\mathcal{M}}([b_1, s_1], .., [b_r, s_r]) = [F^{\mathcal{D}}(b_1, .., b_r), F(s_1, .., s_r)], \tag{ii}$$

$$R^{\mathcal{M}}([b_1, s_1], .., [b_r, s_r]) \;\Leftrightarrow\; R^{\mathcal{D}}(b_1, .., b_r), \tag{iii}$$

$$G_j^{\mathcal{M}}([b_1, s_1], .., [b_r, s_r]) = \begin{cases} [a_i, G_j(s_1, .., s_r)] & \text{if } G_j(s_1, .., s_r) \text{ is } t_i \\ [a_0, G_j(s_1, .., s_r)] & \text{otherwise.} \end{cases} \tag{iv}$$

Since $t_1, .., t_n$ are pairwise different, a term $G_j(\underline{s})$ can equal at most one t_i, and so (iv) is a correct definition. The square brackets in (ii)–(iv) denote pairing. We suppose that this use can be easily distinguished from the situations where they enclose a valuation of variables (and the symbol $\models$ is involved). Let $g : M \to D$ and $h : M \to D$ be the left and right projections, i.e. the functions satisfying $g([b, s]) = b$ and $h([b, s]) = s$. Let $\mathcal{M}^-$ be the reduct of $\mathcal{M}$ to L, i.e. the structure obtained from $\mathcal{M}$ by omitting the realizations of $G_1, .., G_m$. Then it is clear from (ii) and (iii) that g preserves all symbols in L. Thus g is a homomorphism from $\mathcal{M}^-$ to $\mathcal{D}$. Note that in predicate logic without equality a homomorphism does not have to be one to one.

Consider a valuation e in $\mathcal{M}$, a term s and its value $s^{\mathcal{M}}[e]$ in $\mathcal{M}$ with respect to e. The function $g \circ e$ is a valuation in $\mathcal{D}$. If s is a variable, then $s^{\mathcal{M}}[e]$ is $e(s)$ and the equality $g(e(s)) = (g \circ e)(s)$ can be written as $g(s^{\mathcal{M}}[e]) = s^{\mathcal{D}}[g \circ e]$. Using (ii), it is easy to prove that this equality holds for every term s in L. From this and (iii) it follows that $\mathcal{M}^- \models \psi[e] \Leftrightarrow \mathcal{D} \models \psi[g \circ e]$ for every atomic formula ψ in L. The fact that g is onto and another induction show that the latter equivalence holds for every formula ψ in L. Thus *g preserves all formulas in L*. Since $\mathcal{M}^- \models \psi[e]$ is equivalent to $\mathcal{M} \models \psi[e]$ for ψ in L, we have obtained

$$\mathcal{M} \models \psi[e] \Leftrightarrow \mathcal{D} \models \psi[g \circ e] \tag{v}$$

for each formula ψ in L and every valuation e in $\mathcal{M}$. We now define a valuation e_1 in $\mathcal{M}$ as follows:

$$e_1(z) = \begin{cases} [a_i, z] & \text{if } z \text{ is } t_i \\ [e_0(z), z] & \text{otherwise.} \end{cases} \tag{vi}$$

The variables that equal some t_i are $y_1, .., y_k$. Clearly, $g \circ e_1$ and e_0 agree at all other variables. Since $y_1, .., y_k$ are not free in $\Gamma \cup \Delta$ and e_0 satisfies in $\mathcal{D}$ all formulas in Γ and none formula in Δ, it follows from (v) that $\mathcal{M} \models \psi[e_1]$ for every $\psi \in \Gamma$ and $\mathcal{M} \not\models \psi[e_1]$ for every $\psi \in \Delta$. It remains to deal with the formula $\varphi_{\underline{x}}(\underline{t})$.

From (vi) and (ii) it is clear that $h(s^{\mathcal{M}}[e_1]) = s$ for every term s in $L \cup \{G_1, .., G_m\}$. If t_i has the form $G_j(s_1, .., s_r)$, then from (iv) we see that $g(t_i^{\mathcal{M}}[e_1]) = a_i$. If t_i is y_j, then from (vi) we have $g(t_i^{\mathcal{M}}[e_1]) = a_i$ as well. Since $h(t_i^{\mathcal{M}}[e_1]) = t_i$, we have verified that $t_i^{\mathcal{M}}[e_1] = [a_i, t_i]$ for every $i \in \{1, .., n\}$. Then we have:

$$\begin{aligned} \mathcal{M} \models \varphi_{\underline{x}}(\underline{t})[e_1] &\Leftrightarrow \mathcal{M} \models \varphi[e_1(x_1/[a_1, t_1], .., x_n/[a_n, t_n])] \\ &\Leftrightarrow \mathcal{D} \models \varphi[g \circ e_1(x_1/[a_1, t_1], .., x_n/[a_n, t_n])], \end{aligned} \tag{vii}$$

where the first equivalence is an elementary fact about the truth value (w.r.t. a structure and an evaluation) of a formula obtained by substitution, and the second equivalence follows from (v). From (vi) we see that the valuations $g \circ e_1(x_1/[a_1, t_1], .., x_n/[a_n, t_n])$ and $e_0(x_1/a_1, .., x_n/a_n)$ agree at all variables z that are different from all $x_1, .., x_n$ and all $y_1, .., y_k$. They also agree at $x_1, .., x_n$. The remaining variables are those y_j that are not among $x_1, .., x_n$. Since $y_1, .., y_k$ are not free in $\exists \underline{x} \varphi$, those of them that are not among $x_1, .., x_n$ are not free in φ. Thus $g \circ e_1(x_1/[a_1, t_1], .., x_n/[a_n, t_n])$ and $e_0(x_1/a_1, .., x_n/a_n)$ agree at all variables that are free in φ. Then from (i) we have $\mathcal{D} \models \varphi[g \circ e_1(x_1/[a_1, t_1], .., x_n/[a_n, t_n])]$, and (vii) yields $\mathcal{M} \models \varphi_{\underline{x}}(\underline{t})[e_1]$. $\square$

5 An example

We finish by an example on the use of Theorem 1. It is not difficult to verify that in predicate logic without equality, where there are no assumptions about the symbol $=$, the sequent (14) is not logically valid. However, adding $\forall x \forall y(x = y \to G(x) = G(y))^{(\gamma)}$ and $\forall x \forall y(x = y \to (P(x) \to P(y)))^{(\delta)}$ to its antecedent yields a logically valid sequent. Indeed, one can check that these two sentences are everything that is needed to make the informal proof in the beginning of Section 4 gap-free:

> Assume that $P(z)$. From $R(G(G(z)), G(z))$ and β we have $\neg P(G(G(z)))$ and $P(G(z))$. Then $P(z)$ and $P(G(z))$ yield $z = G(z)$ using α. From γ we have $G(z) = G(G(z))$, and then δ yields $P(G(z)) \to P(G(G(z)))$. Since $P(G(z))$, we conclude that $P(G(G(z)))$, which is a contradiction.

This can be translated to a proof $\mathcal{P}$ of $\langle\, R(G(G(z)), G(z)), \gamma; \beta, \alpha, \delta \Rightarrow ; \neg P(z)\,\rangle$. Let the endsequent and thus the entire $\mathcal{P}$ be divided as indicated by the semicolons, let Γ and Π be $\{R(G(G(z)), G(z)), \gamma\}$ and $\{\beta, \alpha, \delta\}$, and put $\Delta = \emptyset$ and $\Lambda = \{\neg P(z)\}$. We have $\mathrm{Symb}(\Gamma, \Delta) = \{R, G, z, =\}$ and $\mathrm{Symb}(\Pi, \Lambda) = \{R, P, z, =\}$. We thus seek a formula μ satisfying $\mathrm{Symb}(\mu) \subseteq \{R, z, =\}$. The proof $\mathcal{P}$ contains no generalizations and in its construction we have a lot of freedom when choosing the order of instantiations. Assume that it ends by two unsubstitutions that yield the sentence β:

$$\frac{\dfrac{\langle\, \Gamma; R(G(G(z)), G(z)) \to \neg P(G(G(z))) \ \&\ P(G(z)), \alpha, \delta \Rightarrow ; \Lambda\,\rangle}{\langle\, \Gamma; \forall y(R(G(G(z)), y) \to \neg P(G(G(z))) \ \&\ P(y)), \alpha, \delta \Rightarrow ; \Lambda\,\rangle}}{\langle\, \Gamma; \forall x \forall y(R(x, y) \to \neg P(x) \ \&\ P(y)), \alpha, \delta \Rightarrow ; \Lambda\,\rangle.} \tag{16}$$

Writing down the entire proof $\mathcal{P}$ and revisiting Section 4, the reader can verify that the procedures described there yield the following formula ν for the upper sequent of (16):

$$R(G(G(z)), G(z)) \ \&\ (z = G(z) \to G(z) = G(G(z))).$$

Notice that the symbol G occurs on both sides of semicolons in the upper sequent of (16) and thus it does not matter that it occurs in ν. Let μ be

$$\exists u \exists v(R(u, v) \ \&\ (z = v \to v = u)).$$

Since $\langle\, \Gamma \Rightarrow \nu\,\rangle$ is logically valid, it is clear that $\langle\, \Gamma \Rightarrow \mu\,\rangle$ is logically valid. Also $\langle\, R(G(G(z)), G(z)) \to \neg P(G(G(z))) \ \&\ P(G(z)), \alpha, \delta, \nu \Rightarrow \Lambda\,\rangle$ is logically valid, and two instantiations applied to it yield $\langle\, \beta, \alpha, \delta, \nu \Rightarrow \Lambda\,\rangle$. The latter sequent is

$$\langle\, \Pi, R(G(G(z)), G(z)) \ \&\ (z = G(z) \to G(z) = G(G(z))) \Rightarrow \Lambda\,\rangle.$$

Now, as G does not occur in $\Pi \cup \Lambda$, Theorem 1 is applicable and yields $\langle\, \Pi, \mu \Rightarrow \Lambda\,\rangle$. Thus the formula μ has the required properties: both $\langle\, \Gamma \Rightarrow \Delta, \mu\,\rangle$ and $\langle\, \Pi, \mu \Rightarrow \Lambda\,\rangle$ are logically valid and we have $\mathrm{Symb}(\mu) \subseteq (\Gamma \cup \Delta) \cap (\Pi \cup \Lambda)$.

6 Comments and conclusions

Let us again consider the situation described in the end of Section 3. Assume that an instantiation rule:

$$\frac{\langle\, \Gamma; \Pi, \theta_z(s) \Rightarrow \Delta; \Lambda\,\rangle}{\langle\, \Gamma; \Pi, \forall z \theta \Rightarrow \Delta; \Lambda\,\rangle} \tag{17}$$

is used in a cut-free proof $\mathcal{P}$ and that we have an interpolant μ of the upper sequent. Then $\mathrm{Symb}(\mu) \subseteq \mathrm{Symb}(\Gamma, \Delta) \cap \mathrm{Symb}(\Pi, \theta_z(s), \Lambda)$ and the sequents $\langle\, \Gamma \Rightarrow \Delta, \mu\,\rangle$

and $\langle\, \Pi, \theta_z(s), \mu \;\Rightarrow\; \Lambda \,\rangle$ are provable (logically valid). The term s and thus also the formula μ can contain function symbols and free variables that do not occur (free) in $\mathrm{Symb}(\Pi, \forall z\theta, \Lambda)$. These symbols are *unwanted* because they must not occur in a possible interpolant of the bottom sequent in (17). If no occurrences of variables in the scope of unwanted function symbols are bound, we can write μ as $\varphi_{\underline{x}}(t_1, \ldots, t_n)$ where the terms $t_1, \ldots, t_n$ are as described in Theorem 1. Then $\langle\, \Gamma \;\Rightarrow\; \Delta, \exists \underline{x}\varphi \,\rangle$ is provable from $\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi_{\underline{x}}(\underline{t}) \,\rangle$ via instantiations, and the provability of $\langle\, \Pi, \forall z\theta, \exists \underline{x}\varphi \;\Rightarrow\; \Lambda \,\rangle$ follows from the provability of $\langle\, \Pi, \forall z\theta, \varphi_{\underline{x}}(\underline{t}) \;\Rightarrow\; \Lambda \,\rangle$ using Theorem 1. Then $\exists \underline{x}\varphi$ is an interpolant of the bottom sequent in (17). However, a problem is that if (17) is not the last inference in the proof $\mathcal{P}$, then symbols that are unwanted at this stage may occur in the scope of function symbols that become unwanted at some later stage. Then getting rid of unwanted symbols (that is, generalizing the terms $t_1, \ldots, t_n$) at this stage introduces bound occurrences of variables, and the just described procedure cannot be simply repeated at later stages.

This explains that Theorem 1 is probably not sufficient to prove the general interpolation theorem for classical predicate logic without equality but with function symbols. It can only solve some cases, as the example in Section 5 suggests.

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