2022 ACTA UNIVERSITATIS CAROLINAE PP. 57–75
PHILOSOPHICA ET HISTORICA 1 / MISCELLANEA LOGICA

SOME NOTES ON EMBEDDINGS, PROJECTIONS, AND EASTON'S LEMMA

ŠÁRKA STEJSKALOVÁ
Dept. of Logic, School of Arts, Charles University in Prague
sarka.stejskalova@ff.cuni.cz

ABSTRACT
We survey some lesser-known facts concerning properties of embeddings and projections between forcing notions. We will also state some generalizations of Easton's lemma. To our knowledge, many of these facts have not been published, so we include their proofs for the benefit of the reader.
Keywords: forcing; forcing notion; dense embedding; regular embedding; complete embedding; projection; chain condition; closure.

1 Introduction

The method of forcing was introduced by Paul Cohen [Coh63, Coh64] in his proof of the independence of the axiom of choice and the continuum hypothesis over ZFC. Since then forcing has proved to be a powerful technique for producing consistency results.

A forcing notion is a partially ordered set $(P, \leq)$ with a greatest element. A substantial part of the forcing machinery deals with combinatorial properties of partially ordered sets. We will survey some results in this area; they are mostly combinatorial and require little knowledge of the forcing method but we do give some more details and definitions in Section 2.

In forcing constructions we often need to compare two forcing notions to find out whether they give rise to the same generic extension or whether one forcing notion gives rise to an extension which is smaller than the other one:

Suppose P and Q are two forcing notions. Does it hold that

> $(*)$ for each P-generic G over V there exists a Q-generic H over V in $V[G]$ such that $V[G] = V[H]$, and conversely?

This question is related to the notion of *forcing equivalence*, which is usually formulated more restrictively than $(*)$, see Definition 3.1. The definition of forcing equivalence is tightly connected to the notion of *dense embedding*. There are several non-equivalent and equivalent definitions of forcing equivalence and some strengthenings which use the notion of dense embedding. We survey some lesser-known facts related to these notions.

A natural weakening of $(*)$ is to ask whether for every P-generic filter G over V, there is a Q-generic filter H over V in $V[G]$, yielding $V[H] \subseteq V[G]$. This question leads to the notions of *complete embedding* and *projection* between forcing notions,

https://www.doi.org/10.14712/24647055.2025.4

functions from P to Q or conversely with some extra properties. Existence of such functions makes it possible to view P as a two-step iteration which starts with Q and is followed by some other forcing notion which we call the quotient forcing (determined by P and Q). In terms of forcing equivalence, P is forcing equivalent to $Q * \dot{R}$, where $\dot{R}$ is a Q-name for the quotient forcing.

In the last section we discuss the chain condition, closure, and distributivity of forcing notions and their preservation by some other forcing notions. We will state some useful variations on Easton's lemma which feature more than two forcing notions and deal with distributivity.

2 Preliminaries

In this section we review some basic facts about forcing and fix notational conventions. The general reference is Jech's book [Jech03]; the treatment of the iteration of forcing notions follows Baumgartner's paper [Bau83].

A forcing notion is a partially ordered set $(P, \leq)$ with a greatest element, which we denote 1_P. To simplify notation, we will often write P instead of $(P, \leq)$ if the ordering is clear from the context.

A condition p is stronger then q, in symbols $p \leq q$, if it carries more information. We say that two condition p and q are compatible, in symbols $p \mid\mid q$, if there is an element of the ordering such that it is below both p and q. We say that they are incompatible, if they are not compatible and we denote this by $p \perp q$. We say that $A \subseteq P$ is an *antichain* if all distinct p, q in A are incompatible; an antichain is *maximal* if every p in P is compatible with some element in A.

If $(P, \leq)$ is a forcing notion, we write $V[P]$ to denote a generic extension by P if the concrete generic filter is not important. Sometimes we write $P \Vdash \varphi$ in place of $1_P \Vdash \varphi$.

We say that $(P, \leq)$ is *separative* if $p \not\leq q$ implies that there is some $r \leq p$ which is incompatible with q. Note that if $(P, \leq)$ is separative, then $p \leq q$ is equivalent to p forcing q into the generic filter.

A forcing notion is said to be *non-trivial* if below every condition there are two incompatible extensions. Otherwise the forcing notion is called *trivial*. Note that if $(P, \leq)$ is non-trivial, then any P-generic filter cannot be an element of the universe.

To obtain all generic extensions it suffices to consider only the separative orders: If $(P, \leq)$ is not separative, then it has a separative quotient which produces the same generic extensions as P. For more details about separative quotients see [Jech03].

Now we define the notion of *a lottery sum* of forcing notions to provide some counterexamples in Section 3. The concept of a "sum" of forcing notions has been around for a long time; for more details see [Ham00].

Definition 2.1 Let $\{ P_i \,;\, i \in I \}$ be an indexed set of forcing notions $(P_i, \leq_{P_i})$. We define *the lottery sum*

$$\bigoplus \{ P_i \,;\, i \in I \} \tag{1}$$

as a forcing notion as follows: The underlying set is $\{ (i, p) \,;\, p \in P_i \ \& \ i \in I \} \cup \{1\}$ where 1 is not an element of $\bigcup \{ P_i \,;\, i \in I \}$, the ordering is such that 1 is the greatest element, and $(i, p) \leq (j, q) \leftrightarrow i = j$ and $p \leq_{P_i} q$.

The intuition is that a $\bigoplus\{ P_i ; i \in I \}$-generic first chooses a forcing notion from $\{ P_i ; i \in I \}$ to force with, and then forces with it.

Finally, we define several forcing notions which we will use to illustrate certain concepts in the following sections.

Cohen forcing is the forcing used by Cohen to show the independence of the continuum hypothesis [Coh63, Coh64].

Definition 2.2 Let $\kappa \geq \omega$ be a regular cardinal and $\alpha > 0$ an ordinal. *Cohen forcing* at κ of length α, denoted by $\mathrm{Add}(\kappa, \alpha)$, is the set of all partial functions from $\kappa \times \alpha$ to 2 of size less than κ. The ordering is by reverse inclusion, i.e. $p \leq q \leftrightarrow q \subseteq p$.

Cohen forcing at κ is κ-closed, and if $\kappa^{<\kappa} = \kappa$, then it is also κ^+-Knaster (see Definition 4.1).

The following forcing was introduced for $\kappa = \omega$ by Sacks in [Sac71] and the generalized version for a regular cardinal $\kappa > \omega$ was introduced by Kanamori [Kan80].

Definition 2.3 Let $\kappa \geq \omega$ be a regular cardinal. We say that a set $(T, \subseteq)$ is a κ-perfect tree if the following hold:

(i) $T \subseteq {}^{<\kappa}2$ and T is closed under initial segments, i.e. if $t \in T$ and $s \in {}^{<\kappa}2$ is such that $s \subseteq t$, then $s \in T$;

(ii) $\forall t{\in}T \exists s{\in}T(t \subseteq s \;\;\&\;\; s^\frown 0 \in T \;\;\&\;\; s^\frown 1 \in T)$, that is, above every node $t \in T$ there is a splitting node s;

(iii) If $\langle s_\alpha \mid \alpha < \gamma \rangle$ for $\gamma < \kappa$ is a $\subseteq$-increasing sequence of nodes in T, then the union $s = \bigcup_{\alpha<\gamma} s_\alpha$ is in T;

(iv) If there are unboundedly many splitting nodes below $s \in T$, then s splits, i.e. if for every $t \subset s$ there exists a splitting node t' such that $t \subset t' \subset s$, then s splits in T.

Note that if $\kappa = \omega$ the items (iii) and (iv) are redundant.

Definition 2.4 Let $\kappa \geq \omega$ be a regular cardinal. *Sacks forcing* at κ, $\mathrm{Sacks}(\kappa, 1)$, is the collection of all κ-perfect trees as in the previous definition. The ordering is by inclusion, i.e. $p \leq q \leftrightarrow p \subseteq q$.

Remark 2.5 For $\kappa > \omega$, we can change the item (iv) in Definition 2.3 in various ways. For example we can require that the item (iv) holds only for nodes of a given fixed cofinality and forbid the splitting on other cofinalities, see [FH12]. Or in general we can require item (iv) only for some stationary subset S of κ; i.e. if there are unboundedly many splitting nodes below $s \in T$ and the height of s is in S, then s splits. We can also add some additional properties regarding the splitting nodes with respect to some stationary subset of κ, see Definition 3.1 (3) in [JS01]. These modifications provide variations of the Sacks forcing with some additional properties.

Now, we define a forcing for adding a closed unbounded subset to a stationary subset of ω_1, which is due to Baumgartner, Harrington and Kleinberg [BHK76].

Definition 2.6 Let $S \subseteq \omega_1$ be stationary. We define a forcing $\mathrm{CU}(S)$ which adds a closed unbounded set to S. The conditions in $\mathrm{CU}(S)$ are closed bounded subset of S ordered by end-extension.

Note that we can define the forcing notion $\mathrm{CU}(X)$ for every subset X of ω_1. However, if X is not stationary, then the forcing $\mathrm{CU}(X)$ collapses cardinals. More precisely, $\mathrm{CU}(S)$ is ω_1-distributive (see Definition 4.1) if and only if S is stationary. If $S \subseteq \omega_1$ is stationary and co-stationary (i.e. $\omega_1 \setminus S$ is stationary) then forcing with $\mathrm{CU}(\omega_1 \setminus S)$ destroys the stationarity of S.

Definition 2.7 Let $\kappa > \omega$ be a regular cardinal. We say that a κ-tree is a *κ-Suslin tree* if it has no cofinal branches and does not contain antichains of size κ.

When forcing with a tree T, the ordering is the reverse ordering of the tree T. A κ-Suslin tree viewed as a forcing notion is κ-cc and κ-distributive (see Definition 4.1), in particular forcing with a Suslin tree preserves all cardinals.

In contrast to the forcing notions mentioned so far, κ-Suslin trees exist only consistently. For example, under $\mathsf{MA}_{\aleph_1}$ (Martin's Axiom) there are no ω_1-Suslin trees; on the other hand, under the assumption of $\diamondsuit$, there are always ω_1-Suslin trees. Sometimes it is convenient to consider Suslin trees with some additional properties:

Definition 2.8 Assume that T is a tree and s is in T. Let T_s denote the set of all nodes in T which are comparable with s; i.e. $T_s = \{\, t \in T\,;\, t \leq_T s \ \vee \ s \leq_T t \,\}$.

Definition 2.9 Let S and T be trees of height ω_1. Let $S \otimes T$ denote the set of all pairs (s, t) such that there is an ordinal $\gamma < \omega_1$ with $s \in S_\gamma$ and $t \in T_\gamma$. The ordering of $S \otimes T$ is component-wise: $(s, t) <_{S \otimes T} (s', t')$ if $s <_S s'$ and $t <_T t'$.

Definition 2.10 Let T be an ω_1-tree and let $0 < n < \omega$. A *derived tree of dimension* n (or an *n-derived tree*) is a tree of the form

$$T_{t_0} \otimes T_{t_1} \otimes \cdots \otimes T_{t_{n-1}}, \tag{2}$$

where $t_0, \ldots, t_{n-1}$ are distinct elements of T of the same height.

A derived tree of dimension 1 is just a tree of the form T_t where $t \in T$.

Definition 2.11 Let $1 \leq n < \omega$. A Suslin tree T is *n-free* if all of its n-derived trees are Suslin. A Suslin tree T is *free* if it is n-free for all $1 \leq n < \omega$.

Free Suslin trees were originally introduced in [Jen] by Jensen under the name full Suslin trees.

Definition 2.12 An ω_1-tree T is *rigid* if there does not exist any automorphism of T other than the identity function. It is *homogeneous* if for all t and s in T with the same height, there exists an automorphism $f : T \to T$ such that $f(t) = s$.

Free ω_1-Suslin trees are rigid. Free and homogeneous ω_1-Suslin trees can be constructed from $\diamondsuit$ (the construction is due to Jensen).

3 Comparing forcing notions

In this section we state some facts concerning the comparison of forcing notions. To our knowledge, many of these facts have not been written up in detail in literature, so we include their proofs for the benefit of the reader. The books [Kun80] and [Abr10] are a general reference for this section.

For the purposes of this section, we assume (unless we say otherwise) that our forcing notions are non-trivial and separative.

3.1 Forcing equivalence and dense embeddings

Recall that if $(Q, \leq_Q)$ is a partial order, then we can find a complete Boolean algebra $(\mathrm{RO}(Q), \leq_{\mathrm{RO}(Q)})$ and a dense embedding i from Q to the positive part $\mathrm{RO}^+(Q)$ of $\mathrm{RO}(Q)$, i.e. to the set $\{\, b \in \mathrm{RO}(Q) \,;\, b > 0_{\mathrm{RO}(Q)} \,\}$. The algebra $\mathrm{RO}(Q)$ is unique up to isomorphism. If $(Q, \leq_Q)$ is in addition separative, then the mapping i is 1-1 and hence it is an isomorphism between Q and some dense subset of $\mathrm{RO}^+(Q)$; in this case we identify Q with a dense subset of $\mathrm{RO}^+(Q)$ when we work with the Boolean completion of Q.

The uniqueness of the Boolean completion can be used to define a natural notion of *forcing equivalence* of forcing notions:

Definition 3.1 We say that two forcing notions $(P, \leq_P)$ and $(Q, \leq_Q)$ are *forcing equivalent* if their Boolean completions are isomorphic.

It is easy to see that forcing-equivalence implies the following weaker model-theoretic property:

> $(*)$ for any P-generic G over V there exists a Q-generic H over V in $V[G]$ such that $V[G] = V[H]$, and conversely.

If P is any forcing notion, then the lottery sum of κ-many copies of P for $\kappa \geq (2^{|P|})^+$ yields a non-equivalent forcing notion which however satisfies the model-theoretic condition $(*)$.

We will discuss several concepts related to the relationship between two forcing notions $(P, \leq_P)$ and $(Q, \leq_Q)$; these concepts will be formulated in terms of the existence of certain functions from P to Q (and conversely) and also in terms of model-theoretic conditions which are weakenings of the condition $(*)$.

Definition 3.2 We say that a function $i : P \to Q$ between partial orders $(P, \leq_P)$ and $(Q, \leq_Q)$ is a *dense embedding* if it is order-preserving, $i(p) \perp i(p')$ whenever $p \perp p'$, and the range of i is dense in Q.

It is easy to check that the existence of a dense embedding implies forcing equivalence, but the converse does not necessarily hold. In fact, we will show below that forcing equivalence does not even imply a weaker condition than the existence of a dense embedding; this weaker condition is stated in Lemma 3.4.

Let us state two lemmas (Lemma 3.3 and 3.4) which are used in practice to check that two forcing notions are equivalent. In both cases, there is a third forcing notion which is used to compare the two. The first lemma provides an equivalent characterization while the second one gives only a sufficient condition. The proofs are an exercise for the reader.

Lemma 3.3 *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be forcing notions. The following are equivalent:*

(i) *$(P, \leq_P)$ and $(Q, \leq_Q)$ are forcing equivalent;*

(ii) *There exists a forcing notion $(S, \leq_S)$ such that both $(P, \leq_P)$ and $(Q, \leq_Q)$ densely embed into $(S, \leq_S)$.*

Instead of P, Q densely embedding into S, we may consider the opposite configuration with S densely embedding into P, Q:

Lemma 3.4 *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be forcing notions. If there exists a forcing notion $(S, \leq_S)$ such that $(S, \leq_S)$ densely embeds into both $(P, \leq_P)$ and $(Q, \leq_Q)$, then the notions $(P, \leq_P)$ and $(Q, \leq_Q)$ are forcing equivalent.*

Notice that Lemma 3.4 gives only a sufficient condition for forcing equivalence. In fact, we will show that the converse of Lemma 3.4 does not hold in general. To find a counterexample, it suffices to consider a forcing notion R with the property that if we force with R, there will be only one generic filter over R in the generic extension by R. More precisely, if G is R-generic over V and H is R-generic in $V[G]$, then $G = H$. If this holds, we say that R has the *unique generic property*.

If R has the unique generic property, then any two disjoint dense subsets P and Q of R will give a counterexample to the converse of Lemma 3.4:

Lemma 3.5 *Let $(R, \leq)$ be a forcing notion*[1] *and let $P, Q \subseteq R$ be two disjoint dense subsets of R. Moreover, assume that there is a forcing notion $(S, \leq_S)$ with dense embeddings $i : S \to P$ and $j : S \to Q$. Then for every $s \in S$ there is a $t \in S$ with $t \leq s$ such that $i(t) \perp j(t)$ in R.*

Proof Let $s \in S$ be arbitrary. Since P and Q are disjoint, we must have $i(s) \neq j(s)$, and in particular $j(s) \not\leq i(s)$ or $i(s) \not\leq j(s)$. Assume without loss of generality that $j(s) \not\leq i(s)$; by separativity, there is $r \in R$ such that $r \leq j(s)$ and $r \perp i(s)$. Since P is dense in R, there is $p \in P$ such that $p \leq r \leq j(s)$. Note that $p \perp i(s)$. Since $p \in P \subseteq R$ and Q is dense in R there is $q \in Q$ such that $q \leq p \leq j(s)$. Since j is a dense embedding, there is $t \leq s$ in S such that $j(t) \leq q \leq p \leq j(s)$. But now $i(t) \leq i(s) \perp p$, and hence $i(t) \perp p$ and $i(t) \perp j(t)$. □

It follows that if P, Q, S, R are as in Lemma 3.5, then R cannot have the unique generic property: If G is S-generic, then $H_0 = i[G]$ and $H_1 = j[G]$ generate two generic filters over R which must be different (the set of the t's with $j(t) \perp i(t)$ is dense in S).

This leaves us with the question whether there is a forcing R with the unique generic property. One well-known example is a 2-free Suslin tree; see Definition 2.11 above for more details. There is also a more complicated example in ZFC, constructed by Jech and Shelah in [JS01] using a variant of the Sacks forcing at an uncountable regular κ.

[1] Recall that we assume that forcing notions are separative.

3.2 Projections, complete embeddings and regular embeddings

Let us now turn to analyzing forcing notions P, Q with P giving a "bigger" extension than Q.

Definition 3.6 We say that a function $\pi : P \to Q$ between $(P, \leq_P)$ and $(Q, \leq_Q)$ is a *projection* if it is order-preserving, $\pi(1_P) = 1_Q$, and

$$\text{for all } p \in P \text{ and all } q \leq_Q \pi(p) \text{ there is } p' \leq_P p \text{ such that } \pi(p') \leq_Q q.^2 \qquad (3)$$

Let π be as above and fix a P-generic filter G. If $D \subseteq Q$ is open dense in Q then $\pi^{-1}{}''D$ is open dense in P and it is easy to see that $\pi''G$ generates a Q-generic filter. Let us denote this generic filter by H.

The forcing P can be decomposed into a two-step iteration of Q followed by a quotient forcing P/H defined as follows:

$$P/H = \{\, p \in P\,;\ \pi(p) \in H \,\}. \qquad (4)$$

Now, it holds that G is a P/H-generic filter over $V[H]$ and $V[G] = V[H][G]$, where in the first model G is taken as a P-generic filter over V and in the second as a P/H-generic filter over $V[H]$.

The converse holds as well. If we first take a Q-generic filter H over V and then a P/H-generic filter G over $V[H]$, then G is a P-generic filter over V and moreover the generic filter H is generated by $\pi''G$.

Definition 3.7 We say that a function $i : Q \to P$ between partial orders $(Q, \leq_Q)$ and $(P, \leq_P)$ is a *complete embedding* if it is order-preserving, $i(q) \perp i(q')$ whenever $q \perp q'$ and

$$\text{for all } p \in P \text{ there is } q \in Q \text{ such that for all } q' \leq q,\ i(q') \mid\mid p. \qquad (5)$$

Analogues of facts mentioned for projections following Definition 3.6 hold also for complete embeddings. Let i be as in the definition above and fix a P-generic filter G. If $D \subseteq Q$ is predense in Q then $i''D$ is predense in P and $i^{-1}{}''G$ is a Q-generic filter. Let us denote this generic filter by H and in $V[H]$ define a quotient forcing as follows:

$$P/H = \{\, p \in P\,;\ \forall q \in H (p \mid\mid i(q)) \,\}. \qquad (6)$$

Then G is a P/H-generic filter over $V[H]$ and $V[G] = V[H][G]$, where in the first model G is taken as a P-generic over V and in the second as a P/H-generic over $V[H]$.

The converse direction holds as well. If we first take a Q-generic filter H over V and define the quotient forcing P/H and then take a P/H-generic filter G over $V[H]$, then G is P-generic over V and moreover the generic filter H is equal to $i^{-1}{}''G$.

Remark 3.8 In general, the quotient forcings (4) and (6) of two separative forcings do not have to be separative. Consider the following easy example using Cohen forcing $\mathrm{Add}(\kappa, \alpha)$ (see Definition 2.2). Let κ be a regular cardinal and $0 < \beta < \alpha$ be ordinals.

---

[2] Note that the condition $\pi(1_P) = 1_Q$ together with (3) ensure that the range of π is dense in Q.

Then it is easy to see that $\pi : \mathrm{Add}(\kappa,\alpha) \to \mathrm{Add}(\kappa,\beta)$ defined by $\pi(p) = p\restriction(\kappa\times\beta)$ is a projection. Let G be an $\mathrm{Add}(\kappa,\beta)$-generic filter over V. Then

$$\mathrm{Add}(\kappa,\alpha)/G = \{\, p \in \mathrm{Add}(\kappa,\alpha)\ ;\ p\restriction(\kappa\times\beta) \in G \,\}. \tag{7}$$

It follows that all conditions in $\mathrm{Add}(\kappa,\beta)$ which are in G are in $\mathrm{Add}(\kappa,\alpha)/G$ and also every condition p in the quotient $\mathrm{Add}(\kappa,\alpha)/G$ is compatible with all conditions in G. Thus two arbitrary conditions $q_0 \neq q_1$ in G witness that $\mathrm{Add}(\kappa,\alpha)/G$ is not separative. This argument can be modified for complete embeddings as well.

Complete embeddings have the following equivalent—and often more useful—characterization.

Definition 3.9 We say that a function $i : Q \to P$ between partial orders $(Q,\leq_Q)$ and $(P,\leq_P)$ is a *regular embedding* if it is order-preserving, $i(q) \perp i(q')$ whenever $q \perp q'$, and $i\,''A$ is a maximal antichain in P, whenever A is a maximal antichain in Q.

Lemma 3.10 *Let $(Q,\leq_Q)$ and $(P,\leq_P)$ be two partial orders. Then a function i from Q to P is a complete embedding if and only if it is a regular embedding.*

Proof Assume that i is a complete embedding from Q into P. Let $A \subseteq Q$ be a maximal antichain and let p in P be given. We will show that there is $a \in A$ such that $i(a) \,||\, p$, hence $i\,''A$ is maximal. As p is in P there is $q \in Q$ such that for all $q' \leq q$, $i(q') \,||\, p$ by (5). Since A is maximal in Q, there is $a \in A$ such that $a \,||\, q$, hence there is $q' \leq q$ such that $q' \leq a$. Therefore $i(q') \leq i(a)$ and $i(q') \,||\, p$. Hence $i(a) \,||\, p$.

For the converse direction assume that i is a regular embedding between Q and P. Let p in P be given and assume for contradiction that for all $q \in Q$ there is a $q' \leq q$ such that $i(q') \perp p$. Then the set

$$D = \{\, q \in Q\ ;\ i(q) \perp p \,\} \tag{8}$$

is dense in Q. Let $A \subseteq D$ be a maximal antichain. Then, by the definition of a regular embedding $i\,''A$ is maximal in P, hence there exists $a \in A$ such that $i(a) \,||\, p$. This is a contradiction as a is also in D and therefore $i(a) \perp p$. □

It would be tempting to claim that a projection from $(P,\leq_P)$ to $(Q,\leq_Q)$ ensures the existence of a complete embedding from $(Q,\leq_Q)$ to $(P,\leq_P)$ and conversely. But in general we need to use the Boolean completions of P and Q.

Lemma 3.11 *Let $(Q,\leq_Q)$ and $(P,\leq_P)$ be two partial orders. Then the following hold:*

(i) If there is a complete embedding from Q to P, then there is a projection from P to $\mathrm{RO}^+(Q)$.

(ii) If there is a projection from P to Q, then there is a complete embedding from Q to $\mathrm{RO}^+(P)$.

Proof (i). Let i be a complete embedding from Q to P. Let us define a function π from P to $\mathrm{RO}^+(Q)$ by

$$\pi(p) = \bigvee\{\, q \in Q \,;\, \forall q' \leq q (i(q') \mid\mid p) \,\}. \tag{9}$$

First note that π is well-defined correctly for all $p \in P$ by (5). Moreover, for all q in Q it holds that

$$\pi(i(q)) = q. \tag{10}$$

To verify (10) denote $Q_p = \{\, q \in Q \,;\, \forall q' \leq q \,(i(q') \mid\mid p) \,\}$ for $p \in P$. Let us first show that $\pi(i(q)) \leq q$, i.e that for all $q^* \in Q_{i(q)}$ it holds that q^* is below q: if not, there exists $q' \leq q^*$, which is incompatible with q, by separativity of Q; however, as i is a complete embedding, it holds that $i(q') \perp i(q)$, which contradicts q^* being in $Q_{i(q)}$. To show $\pi(i(q)) \geq q$, notice that for every $q' \leq q$ it holds that $i(q') \leq i(q)$; therefore q is in $Q_{i(q)}$.

Let us now argue that π is a projection. The order-preservation follows since $Q_{p'} \subseteq Q_p$ whenever $p' \leq p$. Since all conditions are compatible with the condition $1_{\mathrm{RO}^+(Q)}$, we have $\pi(1_P) = 1_{\mathrm{RO}^+(Q)}$.

Let us now prove condition (3). Assume that $b < \pi(p)$ (if $b = \pi(p)$ the condition is satisfied trivially). Since $\pi(p) = \bigvee\{\, q \in Q \,;\, \forall q' \leq q \,(i(q') \mid\mid p) \,\}$, there is $q \in Q$ such that $q \leq b$ and $i(q)$ is compatible with p. Hence there is $p^* \in P$ below both $i(q)$ and p. The rest now follows as $\pi(p^*) \leq \pi(i(q))$ and $\pi(i(q)) = q$ by (10).

(ii). Let π be a projection from P to Q. Let us define a function i from Q to $\mathrm{RO}^+(P)$ by

$$i(q) = \bigvee\{\, p \in P \,;\, \pi(p) \leq q \,\}. \tag{11}$$

First note that i is well-defined for all $q \in Q$ as π is dense. We will show that the function i is a complete embedding. Since $\{p \in P; \pi(p) \leq q'\} \subseteq \{p \in P; \pi(p) \leq q\}$ whenever $q' \leq q$, it is clear that i is order-preserving. Assume that $i(q) \mid\mid i(q')$ for $q, q' \in Q$; we will show that $q \mid\mid q'$. As we work with a complete Boolean algebra, $i(q) \mid\mid i(q')$ is equivalent to:

$$i(q) \wedge i(q') = \bigvee\{\, p \wedge p' \,;\, \pi(p) \leq q \;\&\; \pi(p') \leq q' \,\} \neq 0_{\mathrm{RO}^+(P)}. \tag{12}$$

Therefore there are p and p' in P such that $p \wedge p' \neq 0_{\mathrm{RO}^+(P)}$, $\pi(p) \leq q$ and $\pi(p') \leq q'$. By density of P in $\mathrm{RO}^+(P)$, there is $p^* \in P$ below $p \wedge p'$ and as π is order-preserving, $\pi(p^*)$ is below both q and q'.

To conclude that i is a complete embedding, it suffices by Lemma 3.10 to verify that the image of a maximal antichain is maximal. Let A be a maximal antichain in Q, and $p \in P$ be given (it is enough to consider elements of P as P is dense in $\mathrm{RO}^+(P)$). As A is maximal, there is $a \in A$ such that a and $\pi(p)$ are compatible. Hence there is $q \in Q$ which is below a and $\pi(p)$. By (3), there is $p' \leq p$ such that $\pi(p') \leq q$. Since $i(a) = \bigvee\{\, p \in P \,;\, \pi(p) \leq q \,\}$ and $\pi(p') \leq q$, we conclude $p' \leq i(a)$. Therefore the antichain $i\,''A$ is maximal. □

There is a natural method for defining projections from $(P, \leq_P)$ onto suborders of $(\mathrm{RO}^+(Q), \leq_{\mathrm{RO}^+(Q)})$ in situations in which every P-generic extension $V[G]$ contains a Q-generic filter H.

Lemma 3.12 *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two partial orders. Assume that for every P-generic filter G over V there is in $V[G]$ a Q-generic filter over V. Let $\dot{H}$ be a P-name such that $1_P \Vdash$ "$\dot{H}$ is a $\mathrm{RO}^+(Q)$-generic filter".[3] Then the following hold:*

(i) Define $\pi : P \to \mathrm{RO}^+(Q)$ by

$$\pi(p) = \bigwedge\{\, b \in \mathrm{RO}^+(Q) \,;\, p \Vdash b \in \dot{H} \,\}. \tag{13}$$

Set $b_Q = \pi(1_P) = \bigwedge\{\, b \in \mathrm{RO}^+(Q) \,;\, 1_P \Vdash b \in \dot{H} \,\}$. Let $\mathrm{RO}^+(Q) \restriction b_Q$ denote the partial order $\{\, b \in \mathrm{RO}^+(Q) \,;\, b \leq b_Q \,\}$. Then

$$\pi : P \to \mathrm{RO}^+(Q) \restriction b_Q \text{ is a projection.} \tag{14}$$

(ii) Moreover, π can be defined just using $-Q = \{\, -q \,;\, q \in Q \,\}$:

$$\pi(p) = \bigwedge\{\, -q \,;\, q \in Q \ \&\ p \Vdash -q \in \dot{H} \,\} = \bigwedge\{\, -q \,;\, q \in Q \ \&\ p \Vdash q \notin \dot{H} \,\}. \tag{15}$$

Proof (i). First, we argue that π is well defined, i.e. $\pi(p) > 0_{\mathrm{RO}(Q)}$ for all $p \in P$. To see this, denote:

$$H_p = \{\, b \in \mathrm{RO}^+(Q) \,;\, p \Vdash b \in \dot{H} \,\}. \tag{16}$$

If $\pi(p) = \bigwedge H_p = 0_{\mathrm{RO}^+(Q)}$, then $D = \{\, b \in \mathrm{RO}^+(Q) \,;\, \exists h \in H_p (h \perp b) \,\}$ is dense. Therefore if G contain p, then $H_p \subseteq H = \dot{H}^G$ and also $H \cap D \neq \emptyset$, hence H contains two incompatible elements. This is a contradiction with the assumption that $\dot{H}$ is forced to be an $\mathrm{RO}^+(Q)$-generic filter by P.

Notice also that $\pi(p) = \bigwedge H_p$ is forced by p into $\dot{H}$: Consider the following dense set:

$$D = \{\, b \in \mathrm{RO}^+(Q) \,;\, b \leq \bigwedge H_p \ \vee\ \exists h \in H_p (h \perp b) \,\}. \tag{17}$$

If G contains p, but H does not contain $\bigwedge H_p$, then H must meet D in some element incompatible with some element in H_p. This is a contradiction. Therefore p forces $\pi(p)$ into $\dot{H}$.

Now, we show that π is a projection. The preservation of the ordering is easy. We check condition (3), i.e. for every $p \in P$ and every $c \leq \pi(p)$, there is $p' \leq p$ such that $\pi(p') \leq c$. Let p and c be given. If $c = \pi(p)$, we are trivially done. So suppose $c < \pi(p)$. If for every $p' \leq p$, $p' \nVdash c \in \dot{H}$, then $p \Vdash \pi(p) - c \in \dot{H}$, which contradicts the fact that $\pi(p)$ is the infimum of $H_p = \{\, b \in \mathrm{RO}^+(Q) \,;\, p \Vdash b \in \dot{H} \,\}$. It follows that there is some $p' \leq p$, $p' \Vdash c \in \dot{H}$. Then $\pi(p') \leq c$ as required.

[3] Notice that π defined below depends on the specific name $\dot{H}$ we choose.

(ii). Let p be fixed and let a_p denote $\bigwedge\{-q\,;\, q \in Q \ \&\ p \Vdash -q \in \dot{H}\}$. We wish to show that $\pi(p)$ from (13) is equal to a_p. Clearly $\pi(p) \leq a_p$. For the converse first notice that

$$\pi(p) = \bigwedge\{\, -q\,;\, q \in Q \ \&\ \pi(p) \leq -q \,\}. \tag{18}$$

This follows from the fact that each element b of $\mathrm{RO}^+(Q)$ can be expressed as a supremum of elements of Q which are below b.

Let as denote $\{-q\,;\, q \in Q \ \&\ \pi(p) \leq -q\}$ by $-Q_p$. To conclude the proof it is enough to show that $-Q_p$ is a subset of $\{\, -q\,;\, q \in Q \ \&\ p \Vdash -q \in \dot{H}\,\}$, i.e. to prove that if $\pi(p) \leq -q$ then $p \Vdash -q \in \dot{H}$. However, we already proved that p forces $\pi(p)$ into $\dot{H}$, therefore if $-q \geq \pi(p)$ then $p \Vdash -q \in \dot{H}$. □

Lemma 3.13 *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two partial orders. Assume that for every P-generic filter G over V, there is in $V[G]$ a Q-generic filter over V. Let $\dot{H}$ be a $\mathrm{RO}^+(P)$-name such that $1_{\mathrm{RO}^+(P)} \Vdash$ "$\dot{H}$ is a $\mathrm{RO}^+(Q)$-generic filter".[4] Then the following hold:*

(i) Define $i : \mathrm{RO}^+(Q) \to \mathrm{RO}^+(P)$ by

$$i(b) = \bigvee\{\, a \in \mathrm{RO}^+(P)\,;\, a \Vdash b \in \dot{H}\,\}. \tag{19}$$

Set $b_Q = \bigwedge\{\, b \in \mathrm{RO}^+(Q)\,;\, 1_{\mathrm{RO}^+(P)} \Vdash b \in \dot{H}\,\}$. Let $\mathrm{RO}^+(Q) \restriction b_Q$ denote the partial order $\{\, b \in \mathrm{RO}^+(Q)\,;\, b \leq b_Q\,\}$. Then

$$i : \mathrm{RO}^+(Q) \restriction b_Q \to \mathrm{RO}^+(P) \text{ is a complete embedding,} \tag{20}$$

where (19) implies $i(b_Q) = 1_{\mathrm{RO}^+(P)}$.

(ii) Let $Q \restriction b_Q$ be the partial order $(Q \cap \mathrm{RO}^+(Q) \restriction b_Q) \cup \{b_Q\}$. Then $i' = i \restriction (Q \restriction b_Q)$ from $Q \restriction b_Q$ to $\mathrm{RO}^+(P)$ is a complete embedding.

(iii) Moreover, i' can be defined using only the conditions in P:

$$i'(q) = \bigvee\{\, p \in P\,;\, p \Vdash q \in \dot{H}\,\}. \tag{21}$$

Proof (i). First notice that i is well-defined below b_Q, i.e. for $b \leq b_Q$ the set $\{\, a \in \mathrm{RO}^+(P); a \Vdash b \in \dot{H}\}$ is nonempty. Let us denote this set by $\mathrm{RO}^+(P)_b$. If $b = b_Q$, then $i(b) = 1_{\mathrm{RO}^+(P)}$ by the density argument from (17). Assume that $b < b_Q$. If $\mathrm{RO}^+(P)_b$ is empty, then there is no $a \in \mathrm{RO}^+(P)$ with $a \Vdash b \in \dot{H}$, i.e. $1_{\mathrm{RO}^+(P)} \Vdash b \notin \dot{H}$. Then $1_{\mathrm{RO}^+(P)}$ forces $-b \wedge b_Q$ to be in $\dot{H}$ and this is a contradiction as we defined b_Q to be the infimum of the conditions in $\mathrm{RO}^+(Q)$ which are forced into $\dot{H}$ by $1_{\mathrm{RO}^+(P)}$.

Further notice that $i(b)$ forces b into $\dot{H}$. If not, then there is a below $i(b)$ which forces that b is not in $\dot{H}$ but as a is below $i(b) = \bigvee\{\, a \in \mathrm{RO}^+(P)\,;\, a \Vdash b \in \dot{H}\,\}$, there is $a_0 \leq a$ which forces b into $\dot{H}$. This is a contradiction.

[4] Notice that i defined below depends on the specific name $\dot{H}$ we choose.

If $b \leq b'$, then every $a \in \mathrm{RO}^+(P)$ which forces $b \in \dot{H}$, forces b' in $\dot{H}$ as well, since $\dot{H}$ is forced to be a generic filter, therefore i is order-preserving. The preservation of incompatibility is easy, as compatible conditions cannot force two incompatible conditions into a filter.

To finish the proof, it suffices by Lemma 3.10 to show that the image of a maximal antichan is maximal. Let A be a maximal antichain in $\mathrm{RO}^+(Q)$ and let b in $\mathrm{RO}^+(P)$ be given. As A is a maximal antichain and $\dot{H}$ is forced to be a generic filter, there has to be $a \in A$ and $b' \leq b$ such that $b' \Vdash a \in \dot{H}$. Since $i(a) = \bigvee\{\, b \in \mathrm{RO}^+(P)\ ;\ b \Vdash a \in \dot{H} \,\}$, $b' \leq i(a)$ and hence $b \mid\mid i(a)$; therefore $i\,''A$ is maximal.

(ii). This follows from Lemma 3.16(i).

(iii). Let q be fixed and let a_q denote $\bigvee\{\, p \in P\ ;\ p \Vdash q \in \dot{H} \,\}$. We show that $i(q)$ as in (19) is equal to a_q. Clearly $a_q \leq i(q)$. For the converse, as $i(q)$ is an element of $\mathrm{RO}^+(P)$ and P is dense in $\mathrm{RO}^+(P)$, $i(q) = \bigvee\{\, p \in P\,;\, p \leq i(q) \,\}$; but all conditions below $i(q)$ have to force q in $\dot{H}$, and therefore $i(q) \leq a_q$. □

Remark 3.14 Note that in the previous two lemmas, Lemma 3.12 and Lemma 3.13, we cannot in general require $\pi(1_P) = 1_{\mathrm{RO}^+(Q)}$ or $i(1_Q) = 1_{\mathrm{RO}^+(P)}$, respectively. Consider the lottery sum of $\mathrm{Add}(\aleph_0, 1)$ and $\mathrm{Add}(\aleph_1, 1)$. It is easy to see that every $\mathrm{Add}(\aleph_0, 1)$-generic filter adds a generic filter for the lottery but only below a condition which chooses $\mathrm{Add}(\aleph_0, 1)$.

We conclude this section by further facts about projections and complete embeddings.

Lemma 3.15 *Assume $(P, \leq_P)$ and $(Q, \leq_Q)$ are partial orders and $\pi : P \to Q$ is a projection.*

(i) If P' is dense in P, then $\pi \restriction P' : P' \to Q$ is a projection.

(ii) (a) If P is dense in P', then there is $\pi' \supseteq \pi$ such that $\pi' : P' \to \mathrm{RO}^+(Q)$ is a projection.

(b) If P' is forcing equivalent to P, then there is a projection $\pi' : P' \to \mathrm{RO}^+(Q)$.

(iii) Let $\dot{R}$ be a P-name for a forcing notion. Then π naturally extends to a projection $\pi' : P * \dot{R} \to Q$.

Proof (i). Obvious.

(ii)(a). For $p' \in P'$ define

$$\pi'(p') = \bigvee\{\, \pi(p)\,;\, p \in P \ \&\ p \leq p' \,\}. \tag{22}$$

By density of P in P', $\{\, \pi(p)\ ;\ p \leq p' \,\}$ is non-empty for every p' and therefore $\pi'(p')$ is in $\mathrm{RO}^+(Q)$. If $p' \leq q'$ are in P', then clearly $\pi'(p') \leq \pi'(q')$. Suppose $p' \in P'$ is arbitrary and $b \leq \pi'(p')$. By the definition of $\pi'(p')$, there is $b' \leq b$ such that for some $p \leq p'$, $p \in P$, $b' \leq \pi(p)$. It follows there is some $q \leq p \leq p'$, $q \in P$, such that $\pi(q) = \pi'(q) \leq b' \leq b$ as desired.

(ii)(b). As P is dense in $\mathrm{RO}^+(P)$, by the previous item there is a projection π^* from $\mathrm{RO}^+(P)$ to $\mathrm{RO}^+(Q)$. Since P' is forcing equivalent to P, P' is dense in $\mathrm{RO}^+(P)$, and $\pi' = \pi^* \restriction P'$ is a projection from P' to $\mathrm{RO}^+(Q)$ by (i).

(iii). Define

$$\pi'(p, \dot{r}) = \pi(p), \tag{23}$$

for every $(p, \dot{r})$ in $P * \dot{R}$. If $(p_1, \dot{r}_1) \leq (p_2, \dot{r}_2)$, then in particular $p_1 \leq p_2$, and thus we have $\pi'(p_1, \dot{r}_1) \leq \pi'(p_2, \dot{r}_2)$ because π is order-preserving. If $(p, \dot{r})$ is arbitrary and $b \leq \pi'(p, \dot{r}) = \pi(p)$, then since π is a projection, there is $p' \leq p$ such that $\pi(p') \leq b$. Since $(p', \dot{r}) \leq (p, \dot{r})$, $\pi'(p', \dot{r}) \leq b$ is as required. □

Lemma 3.16 *Assume $(P, \leq_P)$ and $(Q, \leq_Q)$ are partial orders and $i : Q \to P$ is a complete embedding.*

(i) If Q' is dense in Q, then $i \restriction Q' : Q' \to P$ is a complete embedding.

(ii) (a) If Q is dense in Q', then there is an $i' \supseteq i$ such that $i' : Q' \to \mathrm{RO}^+(P)$ is a complete embedding.

(b) If Q' is forcing equivalent to Q, then there exists an $i' : Q' \to \mathrm{RO}^+(P)$ which is a complete embedding.

(iii) Let $\dot{R}$ be a P-name for a forcing notion. Then i naturally extends to a complete embedding $i' : Q \to P * \dot{R}$.

Proof (i). Obvious.

(ii)(a). For $q' \in Q'$ define

$$i'(q') = \bigvee \{\, i(q) \,;\, q \in Q \;\&\; q \leq q' \,\}. \tag{24}$$

By density of Q in Q', $\{\, i(q) \,;\, q \leq q' \,\}$ is non-empty for every q' and therefore $i'(q')$ is in $\mathrm{RO}^+(P)$. If $q'_0 \leq q'_1$ in Q', then clearly $i'(q'_0) \leq i'(q'_1)$.

Assume that $i'(q'_0)$ is compatible with $i'(q'_1)$, then

$$i'(q'_0) \wedge i'(q'_1) = \bigvee \{ i(q_0) \wedge i(q_1) ; q_0, q_1 \in Q \;\&\; q_0 \leq q'_0 \;\&\; q_1 \leq q'_1 \} \neq 0_{\mathrm{RO}(P)}. \tag{25}$$

Therefore there are $q_0 \leq q'_0$ and $q_1 \leq q'_1$ such that $i(q_0)$ and $i(q_1)$ are compatible. By the definition of complete embedding, q_0 is compatible with q_1. Hence $q'_0 \mid\mid q'_1$, as $q_0 \leq q'_1$ and $q_1 \leq q'_1$.

Suppose $b \in \mathrm{RO}^+(P)$ is arbitrary. Then there is $p \in P$, $p \leq b$, by density of P in $\mathrm{RO}^+(P)$. Therefore there is $q \in Q$ so that for all $q^* \in Q$ such that $q^* \leq q$, $i(q^*)$ is compatible with p, hence with b. Now, we need to show that for all $q' \in Q'$ such that $q' \leq q$, $i'(q')$ is compatible with b. Let $q' \leq q$, $q' \in Q'$, be given and denote $Q_{q'} = \{\, i(q) \,;\, q \in Q \;\&\; q \leq q' \,\}$ so that $i'(q') = \bigvee Q_{q'}$. As all conditions in $Q_{q'}$ are compatible with b, and so is $i'(q')$.

(ii)(b). By (a) and the fact that Q is dense in $\mathrm{RO}^+(Q)$ we conclude that there is a complete embedding i^* from $\mathrm{RO}^+(Q)$ to $\mathrm{RO}^+(P)$. Since Q' is forcing equivalent to Q,

Q' is dense in $\mathrm{RO}^+(Q)$, hence $i' = i^* \restriction Q'$ is a complete embedding from Q' to $\mathrm{RO}^+(P)$ by (i).

(iii). Define

$$i'(q) = (i(q), 1_{\dot{R}}). \tag{26}$$

If $q_0 \leq q_1$, then $i'(q_0) = (i(q_0), 1_{\dot{R}}) \leq (i(q_1), 1_{\dot{R}}) = i'(q_1)$ because i is order-preserving. The same argument holds for the preservation of incompatibility. Let $(p, \dot{r})$ be arbitrary. Then there is $q \in Q$ such that for all $q' \leq q$, $i(q') \parallel p$ and therefore for all $q' \leq q$, $i'(q')$ is compatible with $(p, \dot{r})$. □

4 Basic properties of forcing notions

In this section we discuss four basic properties of forcing notions: the chain condition, the Knaster property, closure, and distributivity. We focus on the preservation of these properties by some other forcing notions. Moreover, we state some variations of Easton's lemma which feature more than two forcing notions or deal with distributivity.

Definition 4.1 Let P be a forcing notion and let $\kappa > \aleph_0$ be a regular cardinal. We say that P is:

- *κ-cc* if every antichain of P has size less than κ (we say that P is *ccc* if it is $\aleph_1$-cc).
- *κ-Knaster* if for every $X \subseteq P$ with $|X| = \kappa$ there is $Y \subseteq X$, such that $|Y| = \kappa$ and all elements of Y are pairwise compatible.
- *κ-closed* if every decreasing sequence of conditions in P of size less than κ has a lower bound.
- *κ-distributive* if P does not add new sequences of ordinals of length less than κ.

It is easy to check that all these properties—except for κ-closure—are invariant under forcing equivalence. Regarding closure, note that for every non-trivial forcing notion P which is κ-closed there exists a forcing-equivalent forcing notion which is not even $\aleph_1$-closed (the completion $\mathrm{RO}^+(P)$ is never $\aleph_1$-closed).

Lemma 4.2 *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P is a forcing notion and $\dot{Q}$ is a P-name for a forcing notion. Then the following hold:*

(i) P is κ-closed and P forces $\dot{Q}$ is κ-closed if and only if $P * \dot{Q}$ is κ-closed.

(ii) P is κ-distributive and P forces $\dot{Q}$ is κ-distributive if and only if $P * \dot{Q}$ is κ-distributive.

(iii) P is κ-cc and P forces $\dot{Q}$ is κ-cc if and only if $P * \dot{Q}$ is κ-cc.

Proof The proofs are routine; for more details see [Jech03] or [Kun80]. □

An analogous statement (iii) for the Knaster property is not in general true: it may happen that $P * \dot{Q}$ is κ-Knaster, yet P does not force that $\dot{Q}$ is κ-Knaster. Consider the following example: Assume $\mathsf{MA}_{\aleph_1}$ and let $\dot{Q}$ be an $\mathrm{Add}(\aleph_0, 1)$-name for the $\aleph_1$-Suslin tree added by $\mathrm{Add}(\aleph_0, 1)$ (see Jech [Jech03] for details). Then $\mathrm{Add}(\aleph_0, 1) * \dot{Q}$ is ccc by previous lemma (iii) and as we assume $\mathsf{MA}_{\aleph_1}$, all ccc forcing notions are $\aleph_1$-Knaster. Therefore $\mathrm{Add}(\aleph_0, 1) * \dot{Q}$ is $\aleph_1$-Knaster, but $\mathrm{Add}(\aleph_0, 1)$ forces that $\dot{Q}$ is not $\aleph_1$-Knaster.

If Q is in the ground model, $P * \check{Q}$ is equivalent to $P \times Q$. Let us state some simple properties of products:

Lemma 4.3 *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions. Then the following hold:*

(i) If P and Q are κ-Knaster, then $P \times Q$ is κ-Knaster.

(ii) If P is κ-Knaster and Q is κ-cc, then $P \times Q$ is κ-cc.

Proof The proofs are routine using only combinatorial arguments (a forcing argument is not required). □

Note that in general Lemma 4.3 cannot be strengthened to say that the product of two κ-cc forcing notions is κ-cc (this is called *the productivity of the κ-cc chain condition*): Consider for instance a Suslin tree T at $\aleph_1$ as a forcing notion; then T is $\aleph_1$-cc, but $T \times T$ has an antichain of size $\aleph_1$. A more complicated example can be constructed under CH; this was first done by Laver in unpublished work, see Galvin [Gal80]. Finally note that $\mathsf{MA}_{\aleph_1}$ implies the $\aleph_1$-cc productivity (in fact, it implies that every $\aleph_1$-cc forcing is $\aleph_1$-Knaster) so there is consistently no such example under $\neg\mathsf{CH}$.

These results are specific to the $\aleph_1$-cc and do not extend to cardinals $\kappa^+ > \aleph_1$: it is provable in ZFC that for all cardinals $\kappa \geq \aleph_1$, there is a κ^+-cc forcing whose product is not κ^+-cc. Examples of such forcings were constructed by Todorcevic and Shelah. The most difficult case of the $\aleph_2$-cc was solved by Shelah in 1997, [She97]. For an overview of productivity of the κ-chain condition see [Rin14].

The following lemma summarizes some of the more important forcing properties of a product $P \times Q$ regarding the chain condition.

Lemma 4.4 *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions such that P is κ-Knaster and Q is κ-cc. Then the following hold:*

(i) P forces that Q is κ-cc.

(ii) Q forces that P is κ-Knaster.

Proof (i). This is an easy consequence of Lemmas 4.2(iii) and 4.3(ii).

(ii). We follow the argument from [Cum18], attributed to Magidor. Let $q \in Q$ be a condition which forces that $\{ \dot{p}_\alpha \,;\, \alpha < \kappa \}$ is a subset of P of size κ. For each α choose $q_\alpha \leq q$ which decides the value of $\dot{p}_\alpha$ and denote this value by p_α. Now, by the κ-Knasterness of P, there is $A \subseteq \kappa$ of size κ such that all conditions in $\{ p_\alpha \,;\, \alpha \in A \}$ are pairwise compatible.

Now it suffices to show that there is q_α which forces that $B = \{ \beta \in A \,;\, q_\beta \in \dot{G} \}$ is unbounded in A. Then if G is a generic filter containing q_α, the set $\{ p_\alpha \,;\, \alpha \in B \}$ is a subset of $\{ \dot{p}_\alpha^G \,;\, \alpha < \kappa \}$ of size κ and consists of pairwise compatible conditions.

For contradiction assume that there is no such α. It means that for every $\alpha \in A$ we can find $q_\alpha^* \leq q_\alpha$ and $\gamma_\alpha > \alpha$ such that for all $\beta \geq \gamma_\alpha$

$$q_\alpha^* \Vdash q_\beta \notin \dot{G}. \tag{27}$$

In particular q_α^* is incompatible with all q_β where $\beta \geq \gamma_\alpha$, and therefore also with all q_β^* where $\beta \geq \gamma_\alpha$. Now, it is easy to construct an unbounded subset A^* of A such that all conditions in $\{ q_\alpha^* \,;\, \alpha \in A^* \}$ are pairwise incompatible. This contradicts the assumption that Q is κ-cc. □

Now we mention some properties of the product with respect to the preservation of κ-distributivity and κ-closure. If P and Q are two κ-distributive forcing notions, then the product $P \times Q$ does not have to be κ-distributive. Again consider a Suslin tree T at $\aleph_1$ as a forcing notion: T is $\aleph_1$-distributive (see [Jech03] for the details), but $T \times T$ may[5] collapse $\aleph_1$ and therefore it may not be $\aleph_1$-distributive.[6] See also [DJ74] for a construction of a homogeneous ω_1-Suslin tree whose product collapses ω_1, or [JJ74] for a construction of a rigid ω_1-Suslin tree whose product collapses ω_1. For an example in ZFC, consider a stationary and co-stationary subset S of ω_1. Since S and $\omega_1 \setminus S$ are stationary, both forcing notions $\mathrm{CU}(S)$ and $\mathrm{CU}(\omega_1 \setminus S)$ (see Definition 2.6) are ω_1-distributive. Forcing with $\mathrm{CU}(\kappa \setminus S)$ adds a closed unbounded set to $\mathrm{CU}(\kappa \setminus S)$ and hence S is no longer stationary in the generic extension $V[\mathrm{CU}(\kappa \setminus S)]$ and therefore $\mathrm{CU}(S)$ is not distributive in $V[\mathrm{CU}(\kappa \setminus S)]$.

However, if at least one of P and Q is κ-closed, then the product is κ-distributive. Moreover, if both P and Q are κ-closed, then their product is κ-closed.

The following lemma summarizes some of the important properties of the product $P \times Q$ regarding distributivity and closure.

Lemma 4.5 *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions, where P is κ-closed and Q is κ-distributive. Then the following hold:*

(i) P forces that Q is κ-distributive.

(ii) Q forces that P is κ-closed.

Proof The proof is routine. □

We can also formulate some results for the product of two forcing notions with respect to preservation of chain condition and distributivity at the same time. The following lemma appeared in [Eas70].

Lemma 4.6 (Easton) *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions, where P is κ-cc and Q is κ-closed. Then the following hold:*

[5] As we already mentioned, if T is an $\aleph_1$-Suslin tree, then $T \times T$ is not $\aleph_1$-cc, but it can be $\aleph_1$-distributive. An example of such a tree T is the free $\aleph_1$-Suslin tree.

[6] If P is a forcing notion which is $\aleph_1$-distributive, then P does not collapse $\aleph_1$; the converse does not hold in general. However, if P is a tree of height ω_1, then if P does not collapse $\aleph_1$, it must be $\aleph_1$-distributive.

(i) P forces that Q is κ-distributive.

(ii) Q forces that P is κ-cc.

Proof For the proof of (i), see [Jech03, Lemma 15.19], (ii) is easy. □

Let us make a few comments regarding the limits of Easton's lemma. We cannot strengthen the conclusion in (i) to κ-closure: Consider for instance $P = \mathrm{Add}(\aleph_0, 1)$ and $Q = \mathrm{Add}(\aleph_1, 1)$; it is easy to check that Q is not $\aleph_1$-closed in $V[P]$. Similarly, we cannot weaken in general the assumption that Q is κ-closed to κ-distributivity: If T is an $\aleph_1$-Suslin tree, then T is $\aleph_1$-distributive and ccc and neither of (i) and (ii) holds for T. However, in some cases it suffices to assume that Q is only κ-distributive:

Lemma 4.7 *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions, where P is κ-cc and Q is κ-distributive. Then if Q forces that P is κ-cc, then P forces that Q is κ-distributive.*

Proof Let f be a function from some ordinal $< \kappa$ into ordinals in $V[P][Q]$; we want to show that f is in $V[P]$. Note that $V[P][Q] = V[Q][P]$ and since Q forces that P is κ-cc, f has a nice P-name $\dot{f}$ of size $< \kappa$ in $V[Q]$. Since $\dot{f}$ has size $< \kappa$ and Q is κ-distributive, $\dot{f}$ is already in V and consequently f is in $V[P]$. □

Easton's lemma 4.6 can be generalized in many ways. Let us state one such generalization which combines the chain condition and the closure in a more complicated way (it is probably folklore but we have not found a proof so we give one for the benefit of the reader).

Lemma 4.8 *Let $\kappa > \aleph_0$ be a regular cardinal, let P, R, S be forcing notions and let $\dot{Q}$ be a P-name for a forcing notion. Assume that $P \times R$ is κ-cc and P forces that $\dot{Q}$ is κ-closed. If S is κ-closed, then $(P * \dot{Q}) \times R$ forces that S is κ-distributive.*

Proof Let use denote $(P * \dot{Q}) \times R$ by Z. Assume for simplicity that the greatest condition in $Z \times S$ forces that $\dot{f} : \kappa' \to \mathrm{ORD}$ is a function in $V[Z][S]$ for some fixed $\kappa' < \kappa$ and some name $\dot{f}$. We will find a stronger condition which will force that this function is already in $V[Z]$. As $\dot{f}$ is arbitrary, this will prove the lemma.

By induction in V, we construct sequences $w^\alpha = \langle\, ((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta, s^\alpha_\beta) \mid \beta < \gamma_\alpha < \kappa \,\rangle$ for $\alpha < \kappa'$ of conditions in $Z \times S$ with the following properties:

(i) For each $\beta < \gamma_\alpha$, $w^\alpha_\beta = ((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta, s^\alpha_\beta)$ decides the value of $\dot{f}(\alpha)$;

(ii) 1_P forces that $\langle\, \dot{q}^\alpha_\beta \mid \beta < \gamma_\alpha \,\rangle$ is a decreasing sequence of conditions in $\dot{Q}$;

(iii) The set $\{\, (p^\alpha_\beta, r^\alpha_\beta)\ ;\ \beta < \gamma_\alpha \,\}$ is a maximal antichain in $P \times R$;

(iv) $\langle\, s^\alpha_\beta \mid \beta < \gamma_\alpha \,\rangle$ forms a decreasing sequence in S;

and for α', $\alpha < \alpha' < \kappa'$:

(i) 1_P forces that $\dot{q}^{\alpha'}_0$ is below every $\dot{q}^\alpha_\beta$, $\beta < \gamma_\alpha$;

(ii) $s^{\alpha'}_0$ is below every s^α_β, $\beta < \gamma_\alpha$.

We first construct the sequence w^0 by induction, ensuring as we go the conditions (i)–(iv) above. Choose $w^0_0 = ((p^0_0, \dot{q}^0_0), r^0_0, s^0_0)$ so that it decides the value of $\dot{f}(0)$. Suppose w^0_β has been constructed for every $\beta < \gamma$; we describe the construction of w^0_γ. If γ is a limit ordinal, first take a lower bound of $\langle \dot{q}^0_\beta \mid \beta < \gamma \rangle$ (denote it $\dot{q}'$) and a lower bound of $\langle s^0_\beta \mid \beta < \gamma \rangle$ (denote it s'). This is possible by conditions (ii) and (iv), respectively, and from the assumption that $\dot{Q}$ is forced to be κ-closed and S is κ-closed. If γ is a successor ordinal $\delta + 1$, work with $\dot{q}^0_\delta$ as $\dot{q}'$ and s^0_δ as s'.

If possible, choose a condition $((p, \dot{q}), r, s)$ such that p forces that $\dot{q}$ is below $\dot{q}'$, s is below s', (p, r) is incompatible with all the previous elements (p^0_β, r^0_β), $\beta < \gamma$, and crucially $((p, \dot{q}), r, s)$ decides the value of $\dot{f}(0)$. In more detail, if possible first pick any (p', r') incompatible with all the previous pairs (p^0_β, r^0_β), $\beta < \gamma$. Then by the forcing theorem there must be an extension of $((p', \dot{q}'), r', s')$ which decides the value of $\dot{f}(0)$. We denote this extension $((p, \dot{q}), r, s)$ (note that $p \Vdash \dot{q} \leq \dot{q}'$). Set $w^0_\gamma = ((p, \dot{q}''), r, s)$, where $\dot{q}''$ is a name which interprets as $\dot{q}$ below the condition p, and interprets as $\dot{q}'$ below conditions incompatible with p.

If this is not possible, set $\gamma_0 = \gamma$. Note that $\gamma_0 < \kappa$ since $P \times R$ is κ-cc.

The construction of w^α for $\alpha < \kappa'$ proceeds analogously, while ensuring the conditions (v)–(vi).

By the κ-closure of $\dot{Q}$ and S, we can take a lower bound of all the conditions appearing in the sequences w^α at the coordinates of $\dot{Q}$ and S—denote these lower bounds $\dot{q}$ and s, respectively. Let $G \times F$ be any $Z \times S$-generic containing $((1_P, \dot{q}), 1_R, s)$. We want to argue that we can define $\dot{f}^{G \times F}$ already in $V[G]$. Let $\alpha < \kappa$ be fixed. By the construction there is a unique pair $(p^\alpha_\beta, r^\alpha_\beta)$ such that $((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta)$ is in G. It follows from the construction of the sequences w^α that $\{ (p^\alpha_\beta, r^\alpha_\beta) \ ; \ \beta < \gamma_\alpha \}$ is a maximal antichain in $P \times R$ by condition (iii). Working in $V[G]$, we can define the right value of $\dot{f}(\alpha)$ as the value which is forced by $((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta, s)$. □

References

[Abr10] U. Abraham. Proper forcing. In M. Foreman and A. Kanamori, eds., *Handbook of Set Theory*, vol. I, chap. 5, pp. 333–394. Springer, 2010.

[Bau83] J. E. Baumgartner. Iterated forcing. In A. R. D. Mathias, ed., *Surveys in Set Theory*, vol. 87 of *London Math. Soc. Lecture Note Ser.*, pp. 1–59. Cambridge University Press, 1983.

[BHK76] J. E. Baumgartner, L. A. Harrington, and E. M. Kleinberg. Adding a closed unbounded set. *J. Symb. Logic*, 41(2):481–482, 1976.

[Coh63] P. Cohen. The independence of the continuum hypothesis. *Proc. Nat. Acad. Sci. of the U.S.A.*, 50:1143–1148, 1963.

[Coh64] P. Cohen. The independence of the continuum hypothesis II. *Proc. Nat. Acad. Sci. of the U.S.A.*, 51:105–110, 1964.

[Cum18] J. Cummings. Aronszajn and Kurepa trees. *Archive Math. Logic*, 57(1–2):83--90, 2018.

[DJ74] K. Devlin and H. Johnsbråten. *The Souslin Problem*, vol. 405 of *LNM*. Springer, 1974.

[Eas70] W. B. Easton. Powers of regular cardinals. *Ann. Math. Logic*, 1:139–178, 1970.

[FH12] S.-D. Friedman and R. Honzík. Supercompactness and failures of GCH. *Fundamenta Mathematicae*, 219(1):15–36, 2012.

[Gal80] F. Galvin. Chain conditions and products. *Fundamenta Mathematicae*, 108(1):33–48, 1980.

[Ham00] J. D. Hamkins. The lottery preparation. *Ann. Pure Appl. Logic*, 101(2–3):103--146, 2000.

[Jech03] T. Jech. *Set Theory*. Springer Monographs in Mathematics. Springer, 2003.

[Jen] R. Jensen. The generic Kurepa hypothesis I + II. Unpublished handwritten notes.

[JJ74] R. Jensen and H. Johnsbråten. A new construction of a non-constructible δ^1_3 subset of ω. *Fundamenta Mathematicae*, 81:279–290, 1974.

[JS01] T. Jech and S. Shelah. Simple complete Boolean algebras. *Proc. of the Amer. Math. Soc.*, 129(2):543–549, 2001.

[Kan80] A. Kanamori. Perfect-set forcing for uncountable cardinals. *Ann. Math. Logic*, 19:97–114, 1980.

[Kun80] K. Kunen. *Set Theory: An Introduction to Independence Proofs*. North-Holland, 1980.

[Rin14] A. Rinot. Chain conditions of products, and weakly compact cardinals. *Bull. Symb. Logic*, 20(3):293–314, 2014.

[Sac71] G. E. Sacks. Forcing with perfect closed sets. In *Axiomatic Set Theory (Proc. Sympos. Pure Math., Univ. California, Los Angeles CA, 1967)*, vol. XIII, Part I of *Proc. Symp. Pure Math.*, pp. 331–355. Amer. Math. Soc., Providence RI, 1971.

[She97] S. Shelah. Colouring and non-productivity of $\aleph_2$-c.c. *Ann. Pure Appl. Logic*, 84(2):153–174, 1997.