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THE RABIN–KEISLER THEOREM AND THE SIZES OF ULTRAPOWERS

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ABSTRACT

Recall the Rabin–Keisler theorem which gives a lower bound κ^ω for the size of proper elementary extensions of complete structures of size κ, provided that κ is an infinite cardinal below the first measurable cardinal. We survey—and at places clarify and extend—some facts which connect the Rabin–Keisler theorem, sizes of ultrapowers, combinatorial properties of ultrafilters, and large cardinals.

Keywords: Rabin–Keisler theorem; sizes of ultrapowers; non-regular ultrafilters.

1 Introduction

In this short survey, we gather some facts scattered in the literature which connect first-order theories, elementary extensions and ultrapowers. As a starting point we consider the following question:

> The Löwenheim–Skolem theorem (LS theorem for short) says that every infinite structure M for a language L has an elementary extension of every size greater or equal to $|L| + \aleph_0$. In particular, every theory T with an infinite model has a model of every size greater or equal to $|L| + \aleph_0$. The question is whether the LS theorem really depends on $|L|$, or not.

On a quick look one might think that if M is a countable structure in an uncountable language $L(M)$, then the language must be in some sense "trivial" (except for some countable sublanguage) if it can be realized on a countable domain. This idea might gain more plausibility by the loosely formulated fact that first-order theories are not strong enough to control infinite sizes, so if a theory T has a model of size $\aleph_0$, it probably has models of every infinite size.

We will review below some folkore facts and results which show that this idea is false: the size of the language $|L|$ in the LS theorem is essential, and for example there is consistently a theory whose models exist in every infinite size except for $\aleph_1$ (in fact, this is a consequence of $2^{\aleph_0} = \aleph_2$ as we will see below). The bottom line is that first-order theories can control the sizes of their models provided these sizes are less or equal than the size of $|L| + \aleph_0$ (see the short paper [Mek77] for an example).

The paper is centered around the Rabin–Keisler theorem as stated for instance in [BS74, Theorem 5.6] or [Cha65]. We give this theorem as Theorem 3.11. This theorem marks the importance of the ultrapower construction in the model theory of the

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first-order predicate logic. It is interesting from several perspectives; we will focus on the fact that while it *a priori* does not deal with large cardinals, the very statement of the theorem for an arbitrary κ needs the notion of a measurable cardinal (see Section 4.1). The connection to large cardinals is accentuated by more recent set-theoretic research which shows that the size of ultrapowers is closely connected to combinatorial properties of ultrafilters, which in turn often pre-suppose some large cardinals (see Section 4.3).

These results appeared in various books and papers, but are often written from different perspectives, without proper proofs and with different focus at different times (the results stretch over several decades). We briefly review some of these results using a unified notation with emphasis on the connections to modern set theory and large cardinals. We will also briefly comment on the question whether large cardinals are natural to logic (and mathematics) or they are artificial notions imported by set theory.

2 Proper elementary extensions, ultrafilters generated by "ideal" elements

Recall the standard method of defining an ω_1-complete (normal) ultrafilter on a regular uncountable κ using an elementary embedding (see the reference book [Kan03] for more details and also for the notational conventions): suppose $j : M \to N$ is an elementary embedding between transitive models of set theory M, N (the language is just the language of set theory) such that the critical point of j is a regular cardinal κ and the powerset of κ is a subset of M. Then it is easy to check that

$$U_j = \{ X \subseteq \kappa \, ; \, \kappa \in j(X) \} \tag{1}$$

is an ω_1-complete normal ultrafilter on κ. U_j is generated by "ideal" element κ (in the sense that κ is not in the range of j). Since it is known that such ultrafilters imply consistency of ZFC (and much more), it follows that the existence of $j : M \to N$ as above cannot be proved in ZFC.

It may be surprising that this natural idea of defining an ultrafilter via an "ideal" element can be formulated also in the context of ZFC without any large cardinal strength: see the definition in (2) below. Without large cardinals, the construction will lose its easy formulation, but it is still useful.[1]

Suppose T is a first-order theory in language $L(T)$ with an infinite model. With $L(T)$ given, we write $\lambda_T = |L(T)| + \aleph_0$. By the compactness theorem, it is easy to show that T has models of every size $\geq \lambda_T$. Let us give some observations related to models of size $< \lambda_T$.

We first discuss these notions in the language of structures. The reformulation for theories is discussed in Remark 3.4.

If A is an infinite structure, let $L(A)$ be the language of A.

[1] The reader will note that the LS theorem guarantees the existence of many embeddings like $j : M \to N$ (yielding the elementary extensions of M), but since the first-order logic is weak, it cannot guarantee that N is *well-founded* (equivalently, *transitive*). This looks like a minor thing, but all of the large cardinal strength of U_j comes from this fact.

Definition 2.1 We say that A is a *complete structure* if for every $a \in A$ there is a constant symbol $\dot{a}$ in $L(A)$ with $\dot{a}^A = a$, and for every $R \subseteq A^n$ where $1 \leq n < \omega$ there is a symbol $\dot{R}$ in $L(A)$ of the corresponding arity such that $(\dot{R})^A = R$.

It follows that the language of a complete structure has size $2^{|A|}$.

Suppose A is a complete structure and $\mathrm{ProperExt}(A)$ is the set of proper elementary superstructures of A in $L(A)$, i.e.

$$\mathrm{ProperExt}(A) = \{\, B\,;\; B \text{is in } L(A),\, A \subsetneq B \text{ and } A \prec B \,\}.$$

Let us consider the partial order $\preceq$ on $\mathrm{ProperExt}(A)$. As it turns out, the set of ultrapowers in $\mathrm{ProperExt}(A)$ is dense in the following sense (compare with [Kei71, Theorem 47]):

Theorem 2.2 *Suppose $B \in \mathrm{ProperExt}(A)$. Then there is a non-principal ultrafilter U such that $\mathrm{Ult}(A^A, U) \in \mathrm{ProperExt}(A)$ and $\mathrm{Ult}(A^A, U) \preceq B$ (up to isomorphism). If $|A| = \omega$, then U is uniform.*

Proof Let x be some fixed element in $B \setminus A$. Let us define

$$U = \{\, X \subseteq A\,;\; x \in \dot{X}^B \,\}, \text{where } X = \dot{X}^A. \qquad (2)$$

U contains for every $X \subseteq A$ either X or $A \setminus X$: for $X \subseteq A$ and $Y = A \setminus X$, we have by elementarity that $\dot{X}^B \cup \dot{Y}^B = B$ and $\dot{X}^B \cap \dot{Y}^B = \emptyset$. The other properties of U are verified similarly, and so U is an ultrafilter. Let us consider $\mathrm{Ult}(A^A, U)$ and let us identity $[f_a]$ with a, where f_a is a constant function with value a, so that $A \subseteq \mathrm{Ult}(A^A, U)$.

We cannot in general conclude that U is uniform,[2] but U is always non-principal in the sense that does not contain finite sets: for every finite subset $X = \{x_0, \dots, x_n\}$ of A, there is a first-order formula φ_X which determines the elements of X.[3] It follows by elementarity applied to φ_X that $\dot{X}^B = \dot{X}^A$, and hence $\dot{X}^B$ cannot contain the new element x. If $|A| = \omega$, then it immediately follows that U is uniform.

The fact that U is non-principal implies that $\mathrm{Ult}(A^A, U)$ is a proper elementary extension because the diagonal function $f(a) = a$ is different from every constant function (mod U).

We have verified $A \prec \mathrm{Ult}(A^A, U)$. Let us show $\mathrm{Ult}(A^A, U) \preceq B$. Let us define $h : \mathrm{Ult}(A^A, U) \to B$ by setting, for $\dot{f}^A = f$, $h([\dot{f}^A]) = \dot{f}^B(x)$. We have the following equalities:

$$\begin{aligned} \mathrm{Ult}(A^A, U) \models \varphi([f]) \Leftrightarrow \{\, a \in A\,;\; A \models \varphi(\dot{f}^A(a)) \,\} \in U \Leftrightarrow \\ x \in \dot{X}^B \Leftrightarrow B \models \varphi(\dot{f}^B(x)), \end{aligned} \qquad (3)$$

where $\dot{X}$ is chosen to have $\dot{X}^A = \{a \in A; A \models \varphi(\dot{f}^A(a))\}$. We can take an isomorphic copy if necessary identify $\mathrm{Ult}(A^A, U)$ with an elementary submodel of B. □

[2] In fact, it may not be because $\mathrm{ProperExt}(A)$ can contain small structures which cannot be generated by uniform ultrafilters; see Section 3.4. However, if we are willing to go beyond the first-order logic, we can obtain uniform ultrafilters on uncountable cardinals; see Section 4.2 for more details.

[3] For instance $\varphi_X = (\forall x)(\dot{X}(x) \rightarrow x = \dot{x}_0 \vee \cdots \vee x = \dot{x}_n) \wedge (\dot{X}(\dot{x}_0) \wedge \cdots \wedge \dot{X}(\dot{x}_n))$.

3 Combinatorial properties of ultrafilters and sizes of elementary extensions

3.1 Uniform ultrafilters

It follows by Theorem 2.2 that the minimal size of a structure in $\mathrm{ProperExt}(A)$ for a complete A is determined by the size of ultrapowers. For uniform ultrapowers there are some immediate lower bounds.

Lemma 3.1 *Suppose A is a complete structure of size κ where κ is an infinite cardinal.*

(i) Assume $\kappa^{<\kappa} = \kappa$. Then for every uniform ultrafilter U on κ, $\mathrm{Ult}(A^A, U)$ has size exactly 2^κ.

(ii) Let there exist on κ an almost-disjoint system X of size λ with $\kappa < \lambda \leq 2^\kappa$. Then for every uniform ultrafilter U on κ, $\mathrm{Ult}(A^A, U)$ has size at least λ.

Proof Let us first prove (ii). Let $X = \{ X_i \; ; \; i < \lambda \}$ be an almost disjoint family of size λ of subsets of A (every X_i has size κ and $X_i \cap X_j$ has size $< \kappa$ for $i \neq j$). Let us fix for every i some bijection $f_i : A \to X_i$. It follows that if $i \neq j$, then $\{ a \in A \; ; \; f_i(a) = f_j(a) \}$ is bounded in κ. It follows $[f_i]_U \neq [f_j]_U$, and hence $h(i) = [f_i]_U$ is an injective function from λ to $\mathrm{Ult}(A^A, U)$.

Claim (i) follows from claim (ii) by observing that $\kappa^{<\kappa} = \kappa$ implies that there exists an almost-disjoint system of size 2^κ. □

Lemma 3.1 gives the following (because non-principal equals uniform for ω):

Corollary 3.2 (a version of the Rabin–Keisler theorem) *If A is a complete structure of size ω, then every element of $\mathrm{ProperExt}(A)$ has size at least 2^ω.*

Lemma 3.1 determines the size of ultrapowers via uniform ultrafilters in many situations, for instance under GCH:

Corollary 3.3 *Suppose GCH holds, κ is a regular cardinal, and U is a uniform ultrafilter on A with $|A| = \kappa$. Then $\mathrm{Ult}(A^A, U)$ has size 2^κ.*

However, note that the ultrafilter U from (2) may be non-uniform for uncountable κ, so Lemma 3.1 does not completely determine the least size of structures in $\mathrm{ProperExt}(A)$.

Remark 3.4 Let A be any complete structure. Let T_A be the theory in the language $L(A)$ (including any language A natively has) which contains all sentences which are true in A in this extended language. Note that A is a model T_A, and any other model is up to isomorphism in $\mathrm{ProperExt}(A)$. It follows by Corollary 3.2 that there exists a first-order theory T with language of size 2^ω which has a countable model, and every other model has size at least 2^ω. The theory T may extend ZFC or any other theory as desired.

3.2 Regular ultrafilters

There is a combinatorial concept which is stronger than uniformity and which implies that the associated ultrapower has the maximal size without making the extra assumptions about almost disjoint families and their sizes (as in Lemma 3.1).

Definition 3.5 Let U be an ultrafilter on an infinite cardinal κ. We say that U is *regular* if there is a family $\{ X_i ; i < \kappa \}$ of pairwise distinct sets in U such that every infinite subcollection of $\{ X_i ; i < \kappa \}$ has an empty intersection.

Notice that one can say equivalently that $\{ X_i ; i < \kappa \}$ is a witness for regularity if for every $j < \kappa$ the set

$$Z_j = \{ i < \kappa ; j \in X_i \} \tag{4}$$

is finite. Also note that regularity immediately implies that U is non-principal (does not contain a singleton).

Though it is not immediately clear, regularity implies uniformity:

Lemma 3.6 *Suppose U is a regular ultrafilter on an infinite κ. Then U is uniform.*

Proof Suppose U is regular and suppose for contradiction that U contains some set of size $\mu < \kappa$; let us assume that $\mu \in U$. Let $\{ X_i ; i < \kappa \}$ be some sets in U. We will show that this family doess not witness regularity. Suppose for contradiction it does. Consider the family $\{ X_i \cap \mu ; i < \kappa \}$ which are also sets in U. If this set is of size $< \kappa$, it follows that there is some $X_i \cap \mu$ which is contained as a subset in κ-many X_j's (and their intersection is therefore non-empty because it equals $X_i \cap \mu$), so $\{ X_i ; i < \kappa \}$ does not witnesses regularity. If the set is of size κ, consider for every $\alpha < \mu$ the set Z_α of all $X_i \cap \mu$ which contain α as an element; by our assumption $\{ X_i ; i < \kappa \}$ is a witness for regularity, and so this set must be finite; it follows that $\bigcup_\alpha Z_\alpha$ has size at most μ, but this contradicts the fact that $\bigcup_\alpha Z_\alpha = \{ X_i \cap \mu ; i < \kappa \}$ has size κ. □

We now show that regular ultrafilters give large ultrapowers (we follow [Hod93, Theorem 9.5.4]).

Theorem 3.7 *Let U be a regular ultrafilter over some A of size κ. Then $\mathrm{Ult}(A^A, U)$ has size 2^κ.*

Proof We identify A with κ for easier reading. For every j, let $h_j : {}^{Z_j}A \to A$ be a bijection, where Z_j is as in (4). For every $f : A \to A$, let $f^* : A \to A$ be defined as follows:

$$f^*(j) = h_j(f|Z_j).$$

A disagreement of f and g from A to A on a single argument translates into a disagreement of f^* and g^* on a set in U:

Claim 3.8 *Suppose $f, g : A \to A$ and $f(i) \neq g(i)$ for some $i < \kappa$. Then f^* and g^* are different on all arguments j in X_i (where $X_i \in U$).*

Proof For every Z_j such that $i \in Z_j$ it holds that $f|Z_j \neq g|Z_j$, and since h_j is injective, we have $h_j(f|Z_j) \neq h_j(g|Z_j)$. Now notice that $j \in X_i$ implies $i \in Z_j$ for every j, and so the disagreement of f^* and g^* is witnessed on the whole set $X_i \in U$. □

This shows that if $f \neq g$, then $[f^*]_U \neq [g^*]_U$ because f^* and g^* are different on a set in U. □

Remark 3.9 In fact, the theorem gives information about the size of $\mathrm{Ult}(A^I, U)$ for infinite structure A and regular ultrafilters on I: consider injections $h_j : {}^{Z_j}A \to A$ and functions f from I to A. Then the size of the ultrapower is $|A|^{|I|}$.

However, we do not get a generalization of the Rabin–Keisler theorem regarding the minimal size of structures in $\mathrm{ProperExt}(A)$ because even if all uniform ultrafilters are regular (which holds for instance in $V = L$, or more generally if we forbid some very large cardinals, see Section 4.3 for more details), there are always non-uniform ultrafilters which tend to have small ultrapowers, see Section 3.4.

3.3 Non-σ-complete ultrafilters

We saw that regular ultrafilters are always uniform and that this property makes them not general enough for the analysis of $\mathrm{ProperExt}(A)$. There a different concept, i.e. non-σ-completeness defined below which gives more information.

Definition 3.10 An ultrafilter U on an infinite cardinal κ is called *σ-complete* if it is closed under the intersection of countably many sets in U. The same concept is also called *ω_1-complete*. The extension of this concept to κ-completeness is obvious. U is *non-σ-complete* if it is not σ-complete.

Theorem 3.11 (Rabin–Keisler) *Let κ be an infinite cardinal on which every non-principal ultrafilter is non-σ-complete. If A is a complete structure of size κ, then every element of* $\mathrm{ProperExt}(A)$ *has size at least κ^ω.*

Note that if κ is inaccessible, then $\kappa^\omega = \Sigma_{\nu<\kappa}\nu^\omega = \kappa$, so the theorem does not say much regarding the size of elements in $\mathrm{ProperExt}(A)$ for an inaccessible κ. It has informational value if κ satisfies the assumptions of the theorem and κ is singular of countable cofinality (because in this case $\kappa^\omega > \kappa$), or with failures of GCH which increase the number of countable subsets of κ. For $\kappa = \omega$, it follows directly from an easier construction in Corollary 3.2.

First we show a version of the almost-disjointness property:

Lemma 3.12 *Suppose κ is an infinite cardinal. There is an almost disjoint family X of size κ^ω of countable subsets of κ (for $x \neq y \in X$, $|x \cap y| < \omega$).*

Proof This is a variant of the usual construction of an almost disjoint family of size 2^κ provided $2^{<\kappa} = \kappa$: it is enough to construct an almost disjoint family on $\kappa^{<\omega}$ and then use the bijection between κ and $\kappa^{<\omega}$ to transfer it to κ. On $\kappa^{<\omega}$, the collection of cofinal branches κ^ω through $\kappa^{<\omega}$ viewed as a tree is an example of such a family. □

Let us now prove Theorem 3.11 (following [BS74, Theorem 5.4]):

Proof (Of Theorem 3.11) Suppose $A \prec B$ and B is a proper extension. As in (2), define a non-principal ultrafilter U determined by some fixed element $x \in B \setminus A$. Let

$\langle r_{f,n} \mid f \in {}^{\omega}\kappa, n < \omega \rangle$ be some enumeration of elements of A with respect to some almost disjoint family X from the previous Lemma 3.12: $r_{f,n}$ is the n-th element of the countable subset indexed by f.

Since U is non-σ-complete, there is a strictly decreasing sequence $\langle F_n \mid n < \omega \rangle$ of sets in U with the empty intersection (we may assume $F_0 = A$).

For every $f \in {}^{\omega}\kappa$, let us define a function $\tau_f : A \to A$ as follows

$$\tau_f(a) = r_{f,n}, \text{ iff } a \in F_n \setminus F_{n+1}. \tag{5}$$

Let us write Y_n for $F_n \setminus F_{n+1}$. Clearly $\{ a \in A ; \tau_f(a) = \tau_g(a) \} \notin U$ for $f \neq g$: since f and g are almost disjoint, they can agree only on some finite number n of arguments. By the definition of τ_f and τ_g it follows that $\{ a \in A ; \tau_f(a) = \tau_g(a) \}$ is contained in $Y_0 \cup \cdots \cup Y_n$, and this set is not in U.

This proves that for $f \neq g$, $[\tau_f]_U \neq [\tau_g]_U$, and hence $\mathrm{Ult}(A^A, U)$ and also B have size at least κ^{ω}. □

3.4 Non-uniform ultrafilters

Up to now, we discussed ultrafilters on A which give large ultrapowers of A. We now observe that if we use non-uniform ultrafilters on A, or equivalently ultrapowers of A with uniform ultrafilters on sets smaller than A, we (non-surprisingly) obtain smaller ultrapowers. Let us illustrate this case on the following example:

Lemma 3.13 *Assume* CH. *Suppose U on ω_1 contains some countable set; without loss of generality assume $\omega \in U$. Then $\mathrm{Ult}(\omega_1^{\omega_1}, U)$ has size ω_1.*

Proof Assume for contradiction that there is a family $W = \{ f_\alpha ; \alpha < \omega_2 \}$ of functions from ω_1 into ω_1 which are pairwise U-inequivalent. Since U contains ω, also $W|\omega = \{ f_\alpha|\omega ; \alpha < \omega_2 \}$ must be pairwise U-inequivalent,[4] so in particular pairwise distinct and so $W|\omega$ must have size ω_2. But by CH, $|{}^{\omega}\omega_1| = \omega_1$, a contradiction. □

It follows that the Rabin–Keisler theorem does not directly generalize from ω to ω_1 if we require just the non-principality of the ultrafilters:

Corollary 3.14 *Assume* CH. *Suppose A is a complete structure of size ω_1. Then $\mathrm{ProperExt}(A)$ contains a proper elementary extension of A of the form $\mathrm{Ult}(A^A, U)$ for some U generated by a non-principal ultrafilter on ω_1, and this has size ω_1.*

Proof Let U' be a non-principal ultrafilter on ω. This is a centered system on ω_1 and by Zorn's lemma extends into some non-principal ultrafilter on ω_1. □

Corollary 3.15 *More generally: if A is a complete structure of size κ and $\kappa^{\omega} = \kappa$, then there is a non-principal ultrafilter U on A generated by a countable set such that $\mathrm{Ult}(A^A, U)$ has size κ.*[5]

[4] For every $X \subseteq \omega_1$, $X \in U$ implies $X \cap \omega \in U$ because $\omega \in U$.

[5] Note that [BS74, Theorem 5.1] proves this by taking $\mathrm{Ult}(A^{\omega}, U)$ for a non-principal U on ω. Observing the connection with non-uniform ultrafilters allows one to work just with the ultrafilters on the domain of the structure.

This gives the statement of the full Rabin–Keisler theorem formulated as an equivalence, see for instance [BS74, Theorem 5.1].

4 Some connections with large cardinals

4.1 The limits of the Rabin–Keisler theorem

Theorem 3.11 can be stated with the notion of a measurable cardinal: if κ is the least cardinal with a σ-complete non-principal ultrafilter, then κ is in fact measurable, so the following is true:

Theorem 4.1 (Rabin–Keisler, reformulation) *Suppose κ is an infinite cardinal smaller than the least measurable cardinal. If A is a complete structure of size κ, then every element of* $\mathrm{ProperExt}(A)$ *has size at least κ^ω.*

As we discussed in the paragraph before the statement of Theorem 3.11, the theorem provides a non-trivial lower bound for singular cardinals κ with countable cofinality or in cases with failures of GCH, provided that κ is smaller than the first measurable.

It is a natural question whether the assumptions that κ is smaller than then first measurable, or that there is no σ-complete ultrafilter on κ, are necessary. Surprisingly, not much is known about this problem; in particular the following seems open:

Question 4.2 *Is it consistent that there is a singular cardinal κ with countable cofinality such that for some complete structure A of size κ, there is a proper elementary extension of A of size κ?*

Note the following context for this question: if κ is singular with countable cofinality, no uniform ultrafilter U on κ can be σ-complete. However, it can consistently happen (for instance if there is a strongly compact cardinal) that there is some λ, $\omega < \lambda < \kappa$, some non-principal non-uniform ultrafilter U on κ generated by a set of size λ, and U is σ-complete. Little reflection shows that λ must be greater or equal than the first measurable cardinal. Existence of such U blocks the argument from the proof of Theorem 3.11 because it may be that the ultrafilter from Theorem 3.11 is σ-complete.

4.2 The Rabin–Keisler theorem and strongly compact cardinals

If we are willing to go beyond the first-order logic, then the Rabin–Keisler theorem generalizes to other cardinals.

Let us consider the logic $L_{\kappa,\kappa}$ which allows formulas of length $< \kappa$ with conjunctions and disjunctions of length $< \kappa$ and with quantifications of length $< \kappa$. We say that κ is *strongly compact* if for every theory T in $L_{\kappa,\kappa}$ (in an arbitrarily large signature), if every subtheory of T with size $< \kappa$ has a model, then the whole theory T has a model.

If κ is strongly compact, then it is not difficult to check that the construction in Theorem 2.2 yields a non-principal κ-complete ultrafilter U. The κ-completeness plus non-principality implies that U is uniform, and by Lemma 3.1 and the fact that strong compactness of κ implies $\kappa^{<\kappa} = \kappa$, we know that $\mathrm{Ult}(A^A, U)$ has size 2^κ. It follows we obtain the following theorem:

Theorem 4.3 (Rabin–Keisler, for strongly compact cardinals) *Suppose κ is strongly compact. If A is a complete structure of size κ, then every element of $\mathrm{ProperExt}(A)$ (where elementary extensions are now considered in the infinitary logic $L_{\kappa,\kappa}$) has size at least 2^κ.*

4.3 Non-regular ultrafilters

We saw in Theorem 3.7 that regular ultrafilters on A give the maximal possible size of ultrapowers $\mathrm{Ult}(A^A, U)$. There is a natural question whether there are non-regular ultrafilters; in view of Lemma 3.6 every non-uniform ultrafilter is non-regular, so to avoid trivialities, we are interested in non-regular uniform ultrafilters.

Let us first give a two-parameter version of regularity:

Definition 4.4 Suppose κ is an infinite cardinal, and $\omega \leq \lambda < \mu \leq \kappa$ are cardinals. We say that U is (μ, λ)-*regular* if there μ-many elements $\{ X_i ; i < \mu \}$ from U such that the intersection $\bigcap F$ of any subfamily $F \subseteq \{ X_i ; i < \mu \}$ with $|F| = \lambda$ is empty.

It follows that if U on κ is regular according to Definition 3.5 then it is (κ, ω)-regular. Note that every uniform U on κ is (κ, κ)-regular, so for nontrivial context, λ must be smaller than κ. Lemma 3.5 generalizes as follows:

Lemma 4.5 *Suppose U is a (κ, λ)-regular ultrafilter on an infinite κ, with $\lambda < \kappa$. Then U is uniform.*

Proof This is like the proof of Lemma 3.6 observing that in the second part of the proof, every Z_α has size $< \lambda$, and hence $\bigcup_\alpha Z_\alpha$ has size $< \kappa$, which gives a contradiction. □

The existence of uniform ultrafilters U which are not (κ, λ)-regular for some $\lambda < \kappa$ has a very large consistency strength. On the other hand, it is true in $V = L$ (and other core models) that every uniform ultrafilter is regular. We will not review the relevant results here, but an interested reader can consult [Mag79, FMS88, Don88, SJ99] for more information (ordered chronologically).

For the purposes of this article, let us just comment on the relevance for the Rabin--Keisler theorem. As we mentioned, for uncountable structures, the construction from Theorem 2.2 can yield non-uniform ultrafilters, so there is no direct connection with uniform non-regular ultrafilters. However, we can still ask about the size of the ultrapower $\mathrm{Ult}(A^A, U)$. We saw in Lemma 3.1 that if $|A| = \kappa$, and $\kappa^{<\kappa} = \kappa$, then this ultrapower has alway the maximal size for a uniform U. Not much is known about other possibilities; for instance, the following seems open:

Question 4.6 *Is it consistent that there is a uniform ultrafilter U on ω_1 such that for some complete structure $|A| = \omega_1$, $|\mathrm{Ult}(A^A, U)| < 2^{\omega_1}$?*

Note that for this to happen, U must be non-regular, and it must hold $\omega_1 < 2^\omega < 2^{\omega_1}$ and every almost disjoint family on ω_1 must have size $< 2^{\omega_1}$. By [DD03], a lower bound for the consistency strength of this configuration is an inaccessible stationary limit of measurable cardinals.

4.4 Some more general comments on large cardinals

This statement of Rabin–Keisler theorem, Theorems 3.11 and 4.1, raises a legitimate question regarding the status of measurable cardinals: if there are no measurable cardinals (for example if we assume $V = L$), then the Rabin–Keisler theorem holds for every κ. If measurable cardinals exist, the picture is less clear and not much is known as we already mentioned in the previous sections.

In the interest of simplicity—provided we think that mathematics should be such—it is tempting to assume there are no measurable cardinals. On second thought, this clean cut suffers from various technical drawbacks: forbidding measurable cardinals in V does not by itself remove them from other transitive models of ZFC, so configurations like in Question 4.6 can still arise V even if there are no large cardinals.[6] So for the "clean cut" we should in fact postulate that

$$\text{The theory } \mathsf{ZFC} + \mathbb{M} \text{ is inconsistent,} \tag{6}$$

where $\mathbb{M}$ denotes "there exists a measurable cardinal".

However, this is essentially a finitary statement whose postulation seems arbitrary and without a real mathematical reason. There is an extensive discussion (see for instance [FFMS00]) whether such a reason can be obtained in a weaker sense by considering certain set-theoretic axioms of wide consequence which decide the existence of a measurable cardinal either way. In the context of measurable cardinals, an axiom worth considering could be $V = L$ which implies $\neg\mathbb{M}$, i.e. (6) is weakened to a provable fact

$$\text{The theory } \mathsf{ZFC} + \mathsf{V} = \mathsf{L} + \mathbb{M} \text{ is inconsistent.} \tag{7}$$

Whether $V = L$ is good axiom cannot be decided without a larger context which we have not developed here, and there is no general consensus (see again [FFMS00] for more references and details).

It is equally interesting to ask whether we should postulate

$$\text{The theory } \mathsf{ZFC} + \mathbb{M} \text{ is consistent.} \tag{8}$$

There is the tendency to view this postulate as preferable over the negative (6): unlike (6), (8) can be refuted by a proof of contradiction from $\mathsf{ZFC} + \mathbb{M}$ if there is one, whereas by Gödel's theorem there is no chance to refute (6).[7]

These few comments might suggest that we should not artificially "remove" the problem of measurability from the Rabin–Keisler theorem because there are no real mathematical reasons for doing so.

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[6] The typical "trick" is to start with a universe V where some property like in Question 4.6 holds: If V has no large cardinals, we are done. If it has large cardinals, cut V at the first inaccessible cardinal κ. Then V_κ is a model of ZFC with no large cardinals, and yet the property holds because it concerns only sets low in the cumulative hierarchy.

[7] This is a fine distinction because refuting (8) is the same as verifying (6); but for general methodological reasons it is usually preferable to consider axioms which are in principle refutable over those which can be only verified, but never refuted, if verification is considered unlikely.

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