# ACTA UNIVERSITATIS CAROLINAE
# PHILOSOPHICA ET HISTORICA 1/2022

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# Philosophica et Historica 1/2022

CHARLES UNIVERSITY • KAROLINUM PRESS

Editor: Šárka Stejskalová (Charles University)

https://karolinum.cz/casopis/auc-philosophica-et-historica


ISSN 0567-8293 (Print)
ISSN 2464-7055 (Online)

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2022 ACTA UNIVERSITATIS CAROLINAE PP. 7–25
**PHILOSOPHICA ET HISTORICA 1** / MISCELLANEA LOGICA

# IS THERE ANY LOGIC AT ALL?

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**ABSTRACT**

Though to this point not as popular as logical pluralism or logical monism, logical nihilism has become a serious position in the philosophy of logic and a new alternative in the disputes about the notion of a logical system being right. We will review some basic moments of the discussions that have occurred so far and try to propose a viable version of logical nihilism. Some of the aspects of the doctrine as it has been proposed, mainly by Gillian Russell need revision but overall it proves plausible and well suited in particular to incorporation into inferentialist and expressivist accounts of meaning and logic. From a more general point of view, logical nihilism shows how essential it is to appreciate the pragmatic significance of logic and acknowledge that logical practice bestows its legitimacy on logical theory and not the other way round. Appreciating this, together with lessons about the open-ended nature of meaning of even logical vocabulary, leads to a more dynamic conception of logic.

**Keywords:** logical nihilism; rule generality; determinacy.

When the disputes about logical pluralism and logical monism were in the danger of exhausting their potential, a worthwhile attempt at enriching the debate was undertaken. After many different definitions of these two antagonistic views, a different alternative has been introduced, thus finally exhausting the possibilities of the number of correct logics, namely *logical nihilism*. If it is worthy of consideration that exactly one logic is right or that more logics are right, then why not consider also the option that all logics are wrong? Since its advent, even this new approach has been not only defended but also attacked.

I will present an assessment of this new approach to logic. My goal is to indicate both what I find insightful about logical nihilism but also where its limits lay. Both the defenders and the opponents of logical nihilism have put their views in different frameworks, which partly explains some of the disputes as stemming from misunderstandings. I will try to put it in an inferentialist framework of the Brandomian kind. Both inferentialism and logical nihilism will be explained and defended.

## 1 Virtues and vices of logical nihilism

What is the question that logical nihilism is supposed to provide an answer to? The fact that there are so many logical systems and new ones are continually being developed is in many ways provoking. One would suppose that logic should be particularly fundamental

https://www.doi.org/10.14712/24647055.2025.1

and therewith also particularly certain. And the many logical systems seem to be in conflict with these expected properties. If there are so many possibilities, then perhaps we are not so sure about logical principles. One way to answer this might be to say that only one system is truly logic. Despite appearances, the others are either failed logics or maybe they are fine and describe something rather close to logic but not logic proper. This amounts to logical monism. The other option would be logical pluralism. According to this view, more than one logic is correct. This is then somewhat in tension with the expectations that logic should in particular be fundamental and certain. Maybe these expectation are just wrong or they have to be reinterpreted if we are to accept logical pluralism. And then there is logical nihilism as the option that no logic is correct, after all.

At first sight, nihilism might appear as a merely provocative and of itself rather absurd thesis. When even a classical author such as [Str50, p. 344] closes his *On referring* by remarking that natural language has no exact logic, this thesis is still very controversial in our time, as is shown by the recent criticism by [PS17, pp. 110–111], who accuse Strawson of claiming the obviously absurd thesis that reasoning in natural language is lawless. While this reasoning may be very complicated and perhaps more complicated than even the most sophisticated logical systems would have it, it surely has to follow some laws, if it is to count as reasoning at all. Let us note that Strawson is far from being the last author who has claimed that natural language has no logic or something similar to this position. Thus [Gla15] emphasizes what he considers to be deep differences in nature between natural language and formal logical systems. I will head towards similar conclusions, though from a very different perspective.

But before we consider these matters, let us note that when there was already the debate between logical monism and logical pluralism, logical nihilism had to be considered sooner or later. If there is a dispute about whether just one logic or more logics are right, then obviously the thesis that no logic is right can also be entertained, as outlandish as it might seem to some. And in particular if the whole debate seems problematic, then the failure to consider this extant possible answer to the controversial question would be indefensible.

Any defense of either logical monism or logical pluralism has to say something about logical nihilism, as well, if it is to be plausible. But why might logical nihilism seem hardly worthy of serious consideration? There is an understandable sense that one or another logic has to be right, maybe a logic which has not yet been actually devised by logicians. If there were no such system that was at least possible, then reasoning has no rules and falls prey to the anarchy of arbitrariness. I will argue, though, that this worry is to a large degree misguided. Logical nihilism has a genuine point, yet our reasoning is not arbitrary. Or at least not completely arbitrary, which is an important difference.

### 1.1 Basic arguments for logical nihilism

Later we will see why the mentioned attempt at reductio of logical nihilism does not quite work but now we will look at the arguments which speak for this position. We will look mainly at the arguments of Gillian Russell. But first we should realize that a simple Occam's razor speaks for logical nihilism. This doctrine does not have to defend any logical system and thus claims less problematic theses. On the other hand, it is yet to be shown how lawfulness of reasoning is compatible with no logic being right. As I promised, we will get to this in due course. But now back to the most salient arguments for logical nihilism which have emerged in the discussion so far.

The basic reason introduced by Gillian Russell is that no logical law holds in absolutely all cases. That is, any logical rule of inference fails in some contexts. How can one arrive at such a conclusion? Many logical laws have been an objects of discussion between their opponents and their adherents. The two probably most common and already somewhat hackneyed examples are the law of excluded middle and the explosivity of contradiction. In the case of LEM, adherents of classical logic claim that it is universally valid, while intuitionists claim that its validity is merely restricted, as it does not hold, for example, in the discourse about infinite mathematical objects. And given that logical laws should hold in all cases, LEM does not pass the test for logical laws. Or so the intuitionists say. Similarly, paraconsistent logicians argue that not every contradiction is explosive. Obviously, one could list many other rules of inference which have been an object of similar disputes, for example disputes about the validity of various laws in modal logics.

These historical disputes are commonly exploited by logical pluralists, as they seem to support the claim that one does not have to countenance merely one true logic but that more systems could be legitimate in their own right. There are already two criticisms announcing themselves. First, logical monists can claim that to merit the title of a logical law, the given rule has to be valid in full generality. The possibility of specific rules of inference restricted to a specific domain would hardly shock even Aristotle. What is new and controversial is calling them logic.[1]

Furthermore, there is the idea that laws of inference belong to the very meaning of logical vocabulary. Although the origins of this idea can be traced back at least to Carnap, it gained prominence particularly thanks to Quine [Qui86, p. 80], who claimed of those who were doubting LEM or the explosivity of contradiction that they were just changing the subject. Other authors, such as Peregrin [Per14, pp. 210–213], might be more benevolent but they still draw some line. So while Peregrin may not consider doubts about LEM or explosivity as an unwitting change of topic, his reasoning about modus ponens resembles very much that of Quine about LEM and the explosivity of contradiction.[2] By abandoning modus ponens, Peregrin argues, one is not speaking of a conditional anymore. One could therefore extract a more moderate version of the Quine's position that changing logic really means just changing the subject. Namely, not every rule has to be upheld if we do not want to change the topic but only some rules. Where exactly the line might lie, could be difficult to establish, according to this moderate view. But probably LEM or explosivity of contradiction are somewhat less convincing candidates than modus ponens for the role of the rule which is unshakeable in the sense just indicated. Even in the extreme form offered by Quine, this way of seeing these matters has much to recommend itself but is also too one-sided, even in more tolerant forms such as that I tried to extract from Peregrin. More about this later.

[1] This kind of criticism of logical pluralism can be found in [Pri06, p. 202].
[2] See [Qui86, p. 80] and [Per14, pp. 210–213].

But Gillian Russell tries to push things further and show that even rules which are much more basic can be doubted. If Quine represents one extreme end of the scale by condemning any attempt at questioning a logical law as misguided, so one can consider the opposite extreme. This would be the thesis that every logical rule can be doubted. Russell wants to open up this possibility by attacking exemplars of particularly basic rules. She chooses the identity inference and conjunction elimination as her targets. If even these laws fail to be universally valid, then the induction to all laws in general seems close to warranted. This would mean that every logical law can be doubted and does not hold universally. She formulates her strategy slightly differently in various papers on the topic, yet it will be illustrative to show how she proceeds in [Rus17].

There she countenances the possibility of valuation depending on the position of a given formula. We could imagine a formula which is true when standing in the scope of a binary connective, yet false when standing alone. This would belie the general validity of conjunction elimination. On the other hand, one can also countenance a formula which is evaluated as true when standing among the premises in a sequent, yet false when among the consequences. This would attack the general validity of reflexivity of consequence. We should note, though, that this example is not merely theoretical, as there are substructural logics which part company with reflexivity, just as there are those which part company with weakening, transitivity and other structural features of the consequence relation.[3] Why the failure of reflexivity caused by the introduction of this new formula, called *prem*, does not lead to the empty consequence relation is shown in [Fje21]. Nevertheless, if any law fails to hold in full generality, then no system holds in full generality. And that is enough for Russell for Russell and for her attempt to show that any purported law fails to hold in some contexts and that there is no unshakeable logical principle.

Indeed, if even such seemingly obvious laws as reflexivity or conjunction elimination fail, then no laws can be upheld come what may, as [EG11] proposes alongside Russell. And if we agree with [Pri06] that the point of logic is exactly to find laws of reasoning which hold come what may, then there is really no logic, as [Rus18] advertises already in the title of her article. When pluralists claim that a given law holds only in certain contexts, domains or under some other restricted conditions, Priest sees these restrictions as evidence that the given law is not logical. We have to look, Priest would argue, for those laws which hold unrestrictedly if we are interested in doing logic. While he apparently does not consider the possibility that the set of universally valid logical laws could be empty, he considers this push for generality as a strategy to defend logical monism in the face of logical pluralism. But Russell and Estrada Gonzáles try to use his attack on logical pluralism to undermine his own position and lead us towards logical nihilism. The striving for absolute generality leaves us with empty logical hands.

But now we shall look at some objections to this basic argument based on generality of logical laws. We will go through the individual objections and see to what degree they oblige us to modify logical nihilism to keep it defensible. We will also gradually put elements of inferentialism into play and see how they help us to arrive at a viable overall position.

[3] See [Zar18] for an illustration of what a logic without the reflexivity of consequence can look like.

## 1.2 Criticism—minimalism against nihilism and the point of logic

[Dich20] brings more arguments against logical nihilism. One of them is that by doubting individual logical laws, no matter how fundamental these happen to be, we are still far from showing that no laws hold come what may. He anticipates that this might be seen as mere pedantry, given how impressive the failure of reflexivity of consequence and of elimination of conjunction is. If it were a genuine failure, of course.

Nevertheless, he has a point that this inductive generalization is still not fully conclusive. I will not get into the details of his technical argumentation but in general he argues that relatively minimalistic set of logical laws is still far from being the same as the empty set. Gillian Russel herself acknowledges that although she has shown how to unsettle very fundamental logical laws, this does not mean that no laws at all hold with full generality. Yet, according to her, very weak logic can hardly be of much use and, therefore, minimalism is practically the same as nihilism. She claims that very weak logic can hardly be useful, for instance, for the formalization of reasoning about arithmetic.

Dicher retorts that various logical systems can be designed for variegated purposes and there is no need to relegate logic merely to the purpose of formalizing arithmetical or even mathematical reasoning. There is, Dicher goes on, no independent standpoint from which to assess the usefulness of logic. This might be true, though still his defence of minimalistic logic is too general and weak. Maybe Russell is somewhat too hasty in concluding that minimalistic logic cannot be useful if it does not help very much with formalizing mathematics but still, it is rather up to a defender such as Dicher to show what different purpose an extremely weak logic could have. And this he does not do.

But we do not have to engage, as Dicher would have us, in discussions of whether there could be a possibly not-completely-empty set of laws of reasoning which are extremely weak but hopefully have the virtue of holding come what may. This is because, in fact, logical laws can fail to hold completely come what may and still be of use. And it indeed can still make a very good sense to call them logical. But we will need to present the inferentialist approach to meaning and in particular to the meaning and role of logical vocabulary to appreciate this. This can be done naturally while discussing another objection to Russell's arguments.

*On the legitimacy of counterexamples—real and imaginary monsters*

There are more ways in which one can try to doubt that reflexivity or conjunction elimination have been genuinely undermined by the counterexamples just mentioned. For one, these sentences which change their meaning depending on their position in the argument strike one immediately as unnatural. Clearly, they were devised primarily for the destructive purpose of undermining the aforementioned fundamental logical laws.

Dicher compares the situation to the historical approach to counterexamples to some geometrical laws, which owes its popularity to [Lak76]. Lakatos calls some counterexamples *monsters*, as they, just like the very special formulae of Russell, are very artificial and do little work above just marring the general validity of certain laws. Furthermore, it seems obvious that they were somehow not intended to be covered by the given law and fit it rather surreptitiously. Nevertheless, getting rid of them by merely adjusting the definitions so that they do not fall under it any more, seems ad hoc and not as a role model of sound methodology. Lakatos calls this ad hoc adjustment *monster barring* throughout his book. Dicher anticipates that his proposal not to accept the counterexamples as genuine could be rebuked as logical variety of monster barring. And indeed, Russell sees Dicher's approach as monster barring in logic.

So, if someone said that the sentences Russell proposes are not to be taken as serious counterexamples, would this be an example of illegitimate monster barring, as she is inclined to say? Dicher, on the other hand, claims that it would be rather just imaginary monster barring, meaning that those monsters are merely imaginary. But how can one distinguish between imaginary and real counterexamples in this context? Dicher, I believe, fails to indicate this, as he merely claims about the nihilists that "What they need are actual English sentences having deleterious effects on logical consequence" (p. 6 of [Dich17]). By this he merely signals that formulae such as *prem* seem somehow strange or unnatural to him, which indeed is no strong criterion. And I doubt that a convincing criterion can be found. In fact, Gödel's sentence claiming its own unprovability in a given system is also stretching the natural expectations of what a sentence is. The same holds for the liar sentence and many other exotic exemplars. Dicher goes on to say on p. 7 that these formulae are invented merely to mar the validity of reflexivity and conjunction elimination. Very well, but why should this make them into illegitimate counterexamples? Again, one could say the same about Gödel's sentence, liar sentence and many others. Still, these specific sentences did play a fundamental role in the history of logic. It is therefore hardly clear why Russell's formulae should be treated less indulgently. Russell's counterexamples should therefore be considered as legitimate, insofar as they establish her point that even identity and conjunction elimination are not unshakeable.

*The meaning of logical vocabulary and changing the subject*

But do not all these counterexamples merely change the subject, as Quine had already suspected them of doing? This question is closely related to the question about the meaning of logical vocabulary and whether every law of inference is necessary for the given meaning to be what it is. Here we should remind ourselves that [Qui51] himself in *Two dogmas* contributed to the awareness of how evasive and problematic meaning is. In particular, it is at least problematic to speak of the meaning of an individual expression, as meaning consists primarily in its interrelations with other expressions. It is therefore surprising, to say the least, that Quine puts all this nuance aside and has such a quick answer when it comes to the meanings of logical expressions.

This said, Quine has a point that inference laws indeed import much to what the expressions of a given language mean. Part of my position is constituted by inferentialism, which consists precisely in the thesis that meaning is constituted by inference rules. Therefore, modus ponens definitely has a lot to do with the meaning of conditional, conjunction elimination with the meaning of conjunction and reflexivity of consequence with the notion of deductive reasoning. Nevertheless, what does it mean that a given rule of inference holds? What does it mean that any rule at all holds?

Here we follow [Bra94], who links the rules with normative attitudes of the members of a given normative community. His overall account of the relation between normative attitudes and normative statuses is very complex, particularly as he presents it in his more recent [Bra19]. We will focus mostly on the dependence of normative statuses on normative attitudes. Of many normative statuses, it will be only rules and in particular inference rules which will interest us here. By normative attitudes, I mean primarily our holding of some behavior for right and other for wrong and acting on it. That means encouraging others and possibly oneself to obey the rules and discouraging them from breaching those rules. In this way rules are constituted by our normative attitudes. It is true that the influence goes both ways, as the normative attitudes are also evaluated as right or wrong on the basis of normative statuses and rules that we acknowledge.

Nevertheless, the bottom line is that all rules, including rules of inference, have to be constantly renewed by our normative behavior. And this renewal typically cannot mean just the repetition of the same. By getting into new situations and new contexts, we have to continually reinvent and develop the rules. Rules may also be dropped on some occasions, just as new ones can come into being if we act in the relevant way.

To illustrate how every rule has to be developed, let us think of the classical example of creatures which look like dogs, yet lack lungs. Should scientists discover such an animal, a decision would have to be made, as to whether these are dogs or not. Was the possession of lungs one of the necessary conditions for being a dog? That is hardly firmly established. This situation would force us to make a decision and develop the rule into a new shape, though in a shape which has to be continuous with the previous one. And despite its sci-fi settings, this example is far from being a remote and theoretical possibility but rather an illustration of how all rules behave all the time.

In a similar vein, any rule of inference has to be reinterpreted in new contexts. Is modus ponens an integral part of the conditional? Obviously, if one would claim that no instance of modus ponens holds, such talk would hardly be recognizable as a talk about the conditional. Consequently, it would lack any reasonable sense. Yet that there might occur problematic instances of modus ponens, such as those presented in [McG96], is nothing miraculous. We can both decide that these are genuine counterexamples or that they are illusory as, by the way, McGee himself does after presenting them. Or we can decide one way for some occasions and the other way for others. This is so because we are to a great degree free to choose whether we consider a putative counterexample, e.g., an English sentence containing the words *if-then* as a genuine conditional and thus formalizable by $A \to B$ or not. We can just say that in a given case it merely appears to be conditional but fails to be one, precisely because that would be against modus ponens. But we can also say that it is conditional, though a specific case thereof, and that therewith modus ponens has been shown to fail to hold come what may.

Let us focus once again on the example of formulae which are true when under the scope of a binary connective and untrue outside this scope. Russell uses these formulae to disprove the general validity of conjunction elimination. We are basically free to both acknowledge them as genuine formulae and see them as counterexamples, as well as declare them for non-formulae or do some similar move. Both kinds of moves are needed in our repertory but specific cases do not force us to prefer either above the other. Both Dicher and Russell in the aforementioned controversy assume that only one answer to the question whether the counterexamples are legitimate is possible. Yet there is no correct answer, it is simply up to our free decision to develop the concepts that are discussed. And this is typically the case with rules.[4]

We thus see that even conjunction elimination is a living rule. It is closely related to the very meaning of conjunction and has to be relatively robust if we are to speak of conjunction at all. But how robust exactly, that is never finally established but has to be discussed in the process of our use of conjunction. Conjunction elimination is important for conjunction being what it is and playing an important role in logic. Nevertheless, this rule does not need to be considered valid come what may in order to keep its prominence.

Rules and with them also meanings are always in the making and never get a definite shape for all eternity. Of course there has to be some stability in our rules and therewith also in meanings, but to smaller or greater degree, every rule is always in the process of being established. Furthermore, even though specific laws do belong to the meanings of logical vocabulary, there is in principle nothing against modifying these rules and with them the meanings of logical expressions. If we heed rather the lessons [Qui51] taught us in *Two dogmas*, we see that there is no principled distinction between synthetic statements and analytical ones and therewith also between facts and analysis of meaning. Yes, Quine of *Philosophy of Logic* is right that modifying logic amounts to modifying meaning to some degree. But so does modifying everything else, as is succinctly put by Field:

> On some readings of "differ in meaning", any big difference in theory generates a difference in meaning. On such readings, the connectives do indeed differ in meaning between advocates of the different all-purpose logics, just as 'electron' differs in meaning between Thomson's theory and Rutherford's; but Rutherford's theory disagrees with Thomson's despite this difference in meaning, and it is unclear why we shouldn't say the same thing about alternative all-purpose logics ([Fie09], p. 345).

Thus discussing how broad the validity of modus ponens is can be both described as a discussion about the properties of conditional, considered independent of us as the properties of dogs, as well as a discussion about which definition of the conditional to endorse. Quine has offered an interesting perspective when he showed us that changing the logical laws can be seen as a change of topic. But this perspective is not to be taken as the absolute truth. It can be used well as an argumentation technique. For example, [Per14, pp. 210–213] uses it well against the sceptical doubts of [Bog00] about the validity of modus ponens. Indeed, one can hardly speak of the conditional absolutely independently of modus ponens, so the scepticism partly undermines itself. Nevertheless, this does not mean any talk of the conditional has to be associated only with a one very specific shape of modus ponens and that any discussions about the rule are impossible. Modus ponens is a living rule, just like any other rule.

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[4]Is this heading towards just another form of logical pluralism, after all? If my position should be termed so, then the reader should keep in mind that is it very different from static pluralism of Beall and Restall. While they suppose that more logics are correct in advance and we have to discover which ones, my view countenances logics as being constantly in the making, as living processes.

Let me note that my approach to the meaning of logical vocabulary is a part of broader philosophical theory of meaning in general which acknowledges that meaning is a living and dynamic entity. Among the proponents of this approach I can mention [Rec03], who argues for *contextualism*, i.e., the theory that meaning is always created in a particular context of language use. He goes as far as to claim that this creation cannot be equated with disambiguation, as that would mean choosing from a previously given set of possibilities. Instead, the real use of language is capable of creating possibilities which are altogether new. This approach, in its turn, has a great predecessor in the analysis of rule following in [Wit53].

Furthermore, my approach to the dependence of logical laws on normative attitudes is an application of a broader conception of rules, not only the logical ones. As logical rules do not hold come what may in the sense that no rules are independent of our normative attitudes, so this holds also of all the other rules. Logical nihilism is thus a specific application of the broader doctrine of anti-necessitarianism or possibilism, proposed by [Mor89].

## 2 The meaning of generality

So are there no generally valid rules? In fact, the understanding of rules I just presented entails that there are not. Or rather, it entails that the notion of rules which are valid for ever and ever does not really make sense. It would mean countenancing rules without the supportive normative attitudes. According to the Brandomian analysis of the notion of rule I offered, these would not be rules.

But that seems to force on us the conclusion that all rules are arbitrary, which is unpalatable. To this I want to say two things. First, our rules, even the quite fundamental ones, could have different shapes and the fact that these alternative shapes are difficult to imagine does not make them impossible. Still, any deviation has to be partial to make sense. I have already mentioned that it would be self-defeating to claim that no instance of modus ponens or conjunction elimination is right. Quine illustrates this point very well by adducing the example of trying to subject conjunction exactly to the rules of disjunction of classical logic. Such a modification, Quine rightly notes, would be merely notational.[5] We would not be speaking of an alternative conjunction but just of disjunction. But from this good example he proceeds all too hastily to equate it with the cases when LEM or explosivity of contradiction are being questioned.

In fact, when Quine is speaking of changing the subject, I propose making a distinction between two kinds of changing the subject. In fact, any substantial discussion of any subject changes it but some discussions change it by developing it, while others just jump to another subject without explicitly avowing this, either as a result of being mistaken or with the purpose of deceiving others. Thus speculating about conjunction possibly being subjected to the same rules as disjunction is an example of unwitting jumping, while discussing the possible exceptions to modus ponens is an example of a development, although there can be more legitimate forms of development. In general, though, one cannot decide for every case whether it is a jump or a development.

[5] See [Qui86, p. 81].

These two notions also have to be themselves developed when confronted with their application to specific cases.

[Fra15], in his defense of logical nihilism, claims that both logical monists and pluralists concur in their intuition that logic has to be somehow *out there*. I agree with him and claim that logic is not simply out there but that it is always established anew, not by any arbitrary fiat but rather by continual development. But even logical nihilists could fall prey to something very close to the conviction that logic is out there. They can share in the impression that logic is simply independent of normative attitudes and we can discuss whether there are any fully generally valid logical laws. Out there, according to such approach, there is something that decides which, if any, logical laws hold in full generality. Such a nihilist would then argue that there are none fully valid laws, as all the laws out there are invalidated by some counterexamples. I suspect that Gillian Russell would fall into this category. To this I retort that any such generality is always partially postulated and never simply found in the way we may find facts about the behavior of bears in the mountains of a given region.

And any generality is always relative and partial. Because language is a living entity, there is no way one can legislate about absolutely all occasions of the use of a given expression. Priest, attacking logical pluralism, opposed to the line of thought common among logical pluralists such as [BR06]. These pluralists claim that we can think of the use of a given logic for a certain area of reasoning or domain of objects reasoned about or something similar. Priest retorts that, obviously, reasoning across domains has to be possible:

> Despite the fact that there are relatively independent domains about which we reason, given any two domains, it is always possible that we may be required to reason across domains ([Pri06], p. 204).

I agree. But that does not mean that any rules can be valid come what may in all contexts and areas of reasoning for all eternity. Not only does this fail to happen, it does not make sense due to the intimate connection between rules and normative attitudes.

Priest, accompanied by not a few equally-minded authors, believes that holding come what may should be the very point of logic and the reason why it is important. Logical notions are interrelated with activities such as debating, arguing, denying a thesis, inferring, etc. These activities are quite essential to our rationality and in many ways underlie many more specific activities we engage in. Logic could not be so fundamental if its rules were not very general in comparison with many other rules. And if someone used an expression without many of the laws we take to hold of negation, it would not make any sense to classify this expression as negation. Nevertheless, there is no such thing as fully general validity and for any expression there are still many ways how specifically this expression could be used and some might strike us as exotic. Our own use never pins down just one set of rules.

### 2.1 The natural vs. the formal and logic as an artifact

The problem of logical nihilism is, as I will try to argue now, closely related to the role of artificial formal languages and their relation to the natural language. We have to give the famous quote from Frege, as it is hard to set the stage for the discussion better:

> Das Verhältnis meiner Begriffschrift zu der Sprache des Lebens glaube ich am deutlichsten machen zu können, wenn ich es mit dem des Mikroskops zum Auge vergleiche. Das Letztere hat durch den Umfang seiner Anwendbarkeit, durch die Beweglichkeit, mit der es sich den verschiedensten Umständen anzuschmiegen weiß, eine große Überlegenheit vor dem Mikroskop. Als optischer Apparat betrachtet, zeigt es freilich viele Unvollkommenheiten, die nur in Folge seiner innigen Verbindung mit dem geistigen Leben gewöhnlich unbeachtet bleiben. Sobald aber wissenschaftliche Zwecke große Anforderungen an die Schärfe der Unterscheidung stellen, zeigt sich das Auge als ungenügend. Das Mikroskop hingegen ist gerade solchen Zwecken auf das vollkommenste angepasst, aber eben dadurch für alle andern unbrauchbar. So ist diese Begriffschrift ein für bestimmte wissenschaftliche Zwecke ersonnenes Hilfsmittel, das man nicht deshalb verurtheilen darf, weil es für andere nichts taugt ([Fre79], p. v).[6]

Does this quote support logical nihilism or does it to the contrary show how little value it has? Curiously enough, both positions have been entertained, with Dicher using Frege against logical nihilism and [Cot18] using him to argue for logical nihilism. Cotnoir in fact distinguishes two kinds of nihilism. The first kind claims that natural language lacks any logic, as no rules really hold, while the second kind merely claims that no logic of natural language can be captured by formal languages. He goes on to embrace the second variety of logical nihilism, as he deems it more cautious. His worry might be similar to that we already mentioned, namely that the first kind of logical nihilism would mean declaring reasoning in natural languages lawless, which it obviously is not. He bases his argumentation on pointing to what he considers as fundamental differences between natural and formal languages, for example that formal languages cannot quantify unrestrictedly, have restrictions on expressing their own semantic properties, etc.

Dicher, on the other hand, doubts that it ever was a point of logical systems to capture the logic of natural language. Obviously, much revolves around the problem of how natural and formal reasoning relate to one another. There are two basic approaches to this discussion. One claims that the formal and the natural languages and their reasoning are continuous, the other denies this and claims that there is a lacuna between the two. So which side should we pick and what will it mean for the prospects of logical nihilism? I think both accounts are one-sided and rest on some misconceptions. Let me explain why.

To begin with, the very notion of natural language is rather suspicious, no matter how commonplace it has become. It is not so clear that English, Italian, Chinese, Swahili

[6] English translation: "I believe that I can best make the relation of my ideography to ordinary language clear if I compare it to that which the microscope has to the eye. Because of the range of its possible uses and the versatility with which it can adapt to the most diverse circumstances, the eye is far superior to the microscope. Considered as an optical instrument, to be sure, it exhibits many imperfections, which ordinarily remain unnoticed only on account of its intimate connection with our mental life. But, as soon as scientific goals demand great sharpness of resolution, the eye proves to be insufficient. The microscope, on the other hand, is perfectly suited to precisely such goals, but that is just why it is useless for all others" ([Fre67], p. 6).

and all the languages not definable by a recursive definition of well-formed formulae and maybe an axiomatic system do form a sufficiently homogenous group that can be so readily contrasted with the so-called formal languages. Furthermore, those languages are, in important ways, quite artificial themselves—not everything you hear in the streets of London, New York City or any other place in the English speaking world counts as correct English. The language is regulated, there are codifications of it and so it is quite artificial. [Sha14] calls logic an artifact, which is supposed to differentiate it from natural everyday reasoning. Very well, but English and all the other languages spoken by humans are artefacts in a very similar way.

The term *natural language* is thus itself a somewhat unnatural contrivance of philosophers who use it primarily for contrast and in order to describe what the formal systems are not and how they do not work. Although the quote from Frege is insightful, he wisely expressed himself by means of a metaphor and we should not take it too literally. Basically all the claims of a different nature between natural and formal languages, including the claims by Cotnoir, can be doubted. This is so mainly because you can hardly say much that is definite about the so called natural languages, as they are living entities and not static systems. For example, is the liar sentence formulated in English all right? In what sense? Is it meaningful, does it have a truth value? I do not see why one should expect that the rules for the correct use of English establish any answers to these questions as correct. It just remains open, probably partly because it does not matter for the ordinary use of language.

This is not to say that one cannot point to interesting differences between the so called natural and formal languages, yet it is more problematic than the usual talk about these differences would suggest. And we should not forget that formal languages are strongly dependent on natural languages, as it would be hard to make sense of the validity of the rules in formal logics, if these could not be explained in a language one already understands. That is a further argument against the common understanding of the difference between the two kinds of languages. Obviously, we would not understand how conjunction works in classical or any other logic if we did not understand the word *and* or analogous expressions of English or other languages.

If we take Frege's metaphor very seriously, we can say that logical nihilism might be true but is then relatively trivial. If logical systems do not try to capture the actual rules of reasoning, then it is no wonder that they fail doing it. But what would be the point of devising these artificial systems? In the literature, you can find many commonplace expressions about abstractions, idealizations and similar aspects which we purportedly have to take into consideration when thinking about the relation between everyday reasoning and formal logics. But why abstract, why idealize and do all those things?

Some authors have embraced the view that logical systems are something like models of reasoning. A model shares some salient properties with what it models but may also simplify in other respects, as it is then more easy to handle. For example, a map oversimplifies many things but is practical and tells us a lot about the territory it depicts. Also in physics there are many cases of idealization, as when in Newtonian mechanics we speak of objects moving without experiencing any friction, though we know that this never happens. But I do not see the point of doing similar idealizations or abstractions about reasoning or logical vocabulary. What should we learn about logical vocabulary in this way? We obviously know the logical vocabulary of natural language well enough, as we could not use it otherwise.

The talk about logical systems being models of actual reasoning is at best still underdeveloped. It is another metaphor and I do not believe that it is very illuminating. I want to add that while considering maps of a given territory or movement without friction, we know in principle rather precisely in what sense these models are inaccurate and we can as if subtract their inaccuracies. We know how the distances on the map translate into distances in the depicted territory. We know how to calculate the influence of friction on a given movement. I do not see any similar methods concerning logical systems as models of reasoning or the use of logical vocabulary. Yes, we can say that *and* might, unlike the conjunction of classical propositional logic, express temporal succession. But how does an acquaintance with classical propositional logic help us calculate anything about the actual usage of *and*? What new properties of the expression *and* can we discover by using any logical system?

Rather I find that logical systems mainly show us how actual reasoning does not work and what its rules do not look like. In a way, a map of a given territory also partly shows us what the territory is not like but this is not the best use to make of it and the best kind of lesson to take from it. In the case of logical systems this negative lesson, on the other hand, is more important than the positive lesson of seeing anything new about the actual behavior or usage of logical vocabulary. This is not meant ironically, as I consider this a valuable service. Removing oversimplified preconceptions and prejudices is hardly an easy job and it possesses great value. So, rather than showing us that real reasoning is something like this or that system, the acquaintance with classical logic, intuitionistic logic and all the other systems makes us aware that actual reasoning cannot be captured by any system.

Furthermore, logical systems do show us how our use of logical vocabulary could be modified. Of course, it does not happen that we simply decide to use classical logic or any other system in everyday reasoning. Nevertheless, after getting acquainted with many systems we understand how many possibilities there are as to how to use the logical vocabulary and we can use it with more awareness and not just spontaneously.

### *One more attempt at monster barring*

Having discussed the commonplace division of languages into natural and formal languages, I can react to one more attempt at monster barring of counterexamples to logical laws. When Brouwer, the father of intuitionism, doubted whether the law of excluded middle holds for reasoning in infinitistic mathematics, there is an easy answer a proponent of the law could provide him. Namely, that the validity of LEM is meant for the given formal system. And in this system, it clearly holds come what may. This would actually come very close to what Hilbert as a formalist might have had in mind as an opponent of Brouwer[7].

This answer clearly makes some very good sense but it also has its problems. If it is claimed that every logical law holds only restrictedly, relative to a given formal system, then it could be seen as a variety of logical nihilism. Nevertheless, it would not be a

[7] See [Zach06] which provides the summary of their debates and fights.

very profound variety. Certainly a given system, for example classical or intuitionistic propositional logic, has its autonomy. We play language games with it, for example when students have to solve some problems based on those systems during a logic exam. Nevertheless, a logical system is only in a very stretched sense a language to which we could restrict the validity of a rule. As I have sketched in the previous sections, language as a system of rules has to be upheld by normative attitudes of its users and these attitudes bring with them dynamics which make language living. In this sense, I concur with [Lau14] who describes formal logical systems, as well as programming languages, as borderline cases of languages. This is a view quite contrary to that of the analytical tradition, which might see these systems as the paradigms of real languages.

As I indicated, the formal systems are comprehensible only in the context of our ordinary language. In this sense, they can be seen as part of our natural languages and one which is independent only to a very limited extent. I have argued that natural languages are not very natural and now I add that formal languages are not so much languages. So restricting the validity of some rules of inference, for example of LEM, to the formal systems is problematic. Does this undermine the formalist response to the Brouwerian doubts about a given logical law (it is obvious enough that the same reasoning applies to any logical law, if it applies to LEM)? I think it at least shows that this formalism fails to capture very much of what is interesting about logical laws.

## 3 Logic as a practice

Very often it is assumed that all the nuances of the actual usage of logical vocabulary cannot for some reason be captured by logical theories. I believe this is correct but more needs to be said about why it is the case. Let me try to do this in this last section. We saw that Cotnoir embraces this view and I agree with him, even though, unlike him, I embrace also what he considers to be the stronger and more controversial form of logical nihilism. Namely, that natural language has no logic. At least in the sense that I will now specify.

The first point is that the difference between formal systems and actual reasoning is not merely quantitative. The rules which govern our use of logical vocabulary are not only likely more complicated than those of formal logics, they are also qualitatively different. Yet I think there is a stronger qualitative argument than those of Cotnoir. One way in which I was explicating this qualitative difference was by showing that to be genuine rules, they have to be upheld by our normative attitudes which renders them dynamic. The logical systems, on the other hand, cast an impression that the logical expressions are guided by a definite set of prescriptions valid for all times.

But to illustrate my point differently, I claim that logic is primarily a practice and not a theory. What kind of practice? I think a particularly illuminative answer is given by Robert Brandom [Bra94], who claims that logic makes inference rules explicit. Thus if there is a rule that we can infer a sentence $B$ from sentence $A$, logic enables us to state this rule with the use of the conditional when we say that $A$ implies $B$. Brandom calls his approach to logic *logical expressivism.* Given what I said about the dynamic character of rules, I think that any act of explication partly modifies what it makes explicit. At the very least, it tends to stabilize it and so slow down the dynamics.

The formal logics, on the other hand, are theories and therewith creatures of quite different kind than the logical practice which they come from and to which they relate. What kind of relation do they have? This is very close to asking what the point of doing formal logics is. There can be many points in devising logical systems, not the least of which is that it is intellectually satisfying for logicians. But I think that among these possible points, depicting the rules of actual usage of logical vocabulary ranks very low, if at all. It is true that when Brandomian expressivism is a good idea concerning the extra-logical vocabulary, it can in principle be applied to logical vocabulary as well. That is, if it is useful to make explicit the rules governing the use of extra-logical vocabulary, so it could also make sense to express the rules governing the logical vocabulary. But it is difficult to see what an expression of the tools of expression could bring. And anyway, if there is the difference of kind between logical practice and logical theory, then it is impossible because any depiction would have to be so deeply wrong that it cannot be of any use. But again, it is not clear what the point of it would be, even if it somehow could succeed. I fully agree with Field who, concerning logical pluralism and the possibility of finding different models of behavior of logical vocabulary, remarked:

> But personally I find it hard to get excited about issues related to the extent of indeterminacy in English words ([Fie09], p. 345).

I think that besides indeterminacy, any vicissitudes concerning the actual usage of English words such as *not*, *or*, *some* and *all* hardly make for particularly interesting philosophical issues. The same, of course, holds for translations of these words in all the other languages spoken. And competent language users should be aware of the indeterminacies that there might be. Maybe the very fact that we can contrive various logical systems and see their inadequacy makes us more acutely aware of these nuances and the potential dangers of misunderstanding or manipulation that might lie in them.

Furthermore, various attempts at loosening this relation between logical theory and logical practice are not of much avail. For example, according to [Coo10] logical theories should be seen not as direct representations but rather just as models of reasoning. [PS17] go further by claiming that we search for reflective equilibrium between theory and practice. While theories provide models which might not exactly correspond to the actual usage of logical vocabulary or actual reasoning, they are more easy to handle. But why would we look for the equilibrium? This view that logical systems are models of actual usage of logical vocabulary is supposed to foster an analogy between logic and, for instance, physics. But while physical theories enable us to say something interesting and new about the reality they model, I doubt that logical theories do anything comparable. Of course, logicians engage in intellectually challenging and fascinating enterprises when they study whether a given semantics is axiomatizable, whether a given calculus is decidable, etc. But these are still questions pertaining merely to the alleged models, not to what they are supposed to model.

And even if there were some pragmatic use to finding a reflective equilibrium between logical systems as models and the reality they model, how can we make sense of applying the idea of reflective equilibrium here? And are there any ways to measure whether formal logic is somehow approaching the state of reflective equilibrium with actual reasoning? Or do only some formal logics come close to it? All these explanations are themselves too much on the theoretical side and too little on the practical side of the aisle. All these doctrines revolve around the idea that logical systems have to represent something, no matter how watered down this idea might come to be by invoking models and the like. And that is why logical nihilism is quite a good answer to them.

Regarding logic primarily as a practice also disarms the possible objection to logical nihilism that it undermines itself. The objection claims that while logical nihilism, just as any other thesis, has to be supported by some reasoning, it precludes any reasoning whatsoever. This objection sees in logical nihilism a variety of scepticism. But my account, which can be overall seen as a variety of logical nihilism, does not undermine reasoning but only shows that its basis is not theoretical. If scepticism is doomed to undermine itself, then my version of logical nihilism is not sceptical.

## 4 Logical nihilism—some final disambiguation and a final verdict

There is an interesting proposal which goes into the direction of logical nihilism, namely so called logical particularism, as it has been presented in [PW18]. This is the thesis that a specific logical system can be applied only to a limited extent, for example only when we reason about a specific topic or a specific domain. In fact, the logical pluralism of Beall and Restall can be seen as a specific variety of logical particularism.

I share the view that no logical laws can be said to be valid in all discourse come what may. This is still practically identical with logical particularism. But it is possible to go further. In fact, one should go further. Recall the discussion about putative counterexamples to a given logical law, for example modus ponens. We are free to both acknowledge the counterexamples or to proclaim them as illusory, as we can interpret them as not really containing conditional, precisely because that would violate modus ponens. But this means that we cannot claim that any logical law simply holds or does not hold no matter how much we restrict the area in which we discuss its validity. No matter how much we particularize the validity, there is no matter of fact which forces us to either refute or acknowledge it. Logical particularism is thus too weak.

And although this view is closely related to the appreciation of the dynamic nature of language and reasoning which constantly develop, this does not mean that we can decide questions about the validity of a given law even when we restrict ourselves to a specific slice of time. Not only does it not make sense to proclaim a given law for valid or invalid diachronically, it is the same synchronically. But this is not much more than just the application of the lessons of *Two dogmas of empiricism*, namely that any statement, including a statement about a logical law, can be both upheld or sacrificed in the face of theoretical hardships. And honouring further lessons from [Qui60] and also from [Dav73], we see that determinacy indeed begins at home, even in logic.

Indeed, logical practice is determinate enough for its purposes. It is not the case that anything goes, I do not herald any form of logical anarchy. Clearly, a given logical system has to be somehow similar to the actual logical practice, if it is to be counted as a logical system at all. Yet, there is nothing out there which determines which logical laws hold and which logical systems are correct.

### 4.1 The verdict

Should we be logical nihilists, then? We saw that, just as with logical monism and logical pluralism, logical nihilism can be and actually has been spelled out in more ways. My point is that logic is a practice and as such is governed by rules which are by their nature live and dynamic. There is therefore no definite shape they have and therefore also no way to capture them correctly. In this sense logical nihilism is correct, as no logic is exactly right. More fundamentally, though, it is wrong to think that they should try to be right. Indeed, I could strengthen the attack and claim that the idea of any logical system being right is not only false but makes no good sense when analyzed.

Logical nihilism proves to be not only a defensible but actually a very reasonable position. Nevertheless, spelling it out requires a revision of or at least going beyond the form in which logical nihilism is typically defended, namely that due to Gillian Russell. We should overcome the supposition that there is some kind of a fact as to how many logics are correct. Thinking about logical systems in that way invites misguided questions. And not only that it fails to be of much real interest, it is based on wrong conception of rules and normativity.

As far as the so often discussed generality of logic is concerned, I have argued that it is not in itself a particularly important feature. True, the logical practice of making inferential relations explicit, i.e. what logic does according to Brandom's logical expressivism, is itself quite universal and can be applied to all kinds of discourse. In this sense, logic indeed is general. But it is general as a practice, not as a set of specific laws which would be valid for all eternity. This is not a vice but a significant virtue because the logical practice has to be itself dynamic if it is supposed to be of any use in making explicit the other living conceptual practices.

**Acknowledgement** This work was supported by the grant project of The Czech Science Foundation (GAČR), number *23-07119S* with the title *Meaning as an object—principles of semantic theories*. The project is led by Jaroslav Peregrin from the Institute of Philosophy of the Czech Academy of Sciences.

## References

[Bog00] P. Boghossian. Knowledge of logic. In P. Boghossian and C. Peacocke, eds., *New Essays on the A Priori*, pp. 229–254. Clarendon Press, London, 2000.

[BR06] J. Beall and G. Restall. *Logical Pluralism*. Oxford University Press, Oxford, 2006.

[Bra94] R. Brandom. *Making it Explicit*. Harvard University Press, Cambridge MA, 1994.

[Bra19] R. Brandom. *A Spirit of Trust: A Reading of Hegel's Phenomenology*. Harvard University Press, Cambridge MA, 2019.

[Coo10] R. Cook. Vagueness and mathematical precision. *Mind*, 111(442):225–247, 2010.

[Cot18] A. J. Cotnoir. Logical nihilism. In Wyatt et al. [WPK18], pp. 301–329.

[Dav73] D. Davidson. Radical interpretation. *Dialectica*, 27(1):314–328, 1973.

[Dich17] B. Dicher. A proof-theoretic defense of meaning-invariant logical pluralism. *Mind*, 125(499):727–757, 2017.

[Dich20] B. Dicher. Requiem for logical nihilism, or: Logical nihilism annihilated. *Synthese*, 197(1):1–24, 2020.

[EG11] L. Estrada Gonzáles. On the meaning of connectives (apropos of a non-necessitarianist challenge). *Logica Universalis*, 5:115–126, 2011.

[Fie09] H. Field. Pluralism in logic. *Review of Symb. Logic*, 2(2):342–359, 2009.

[Fje21] A. Fjellstad. Logical nihilism and the logic of prem. *Logic and Logical Philosophy*, 30(1):311–325, 2021.

[Fra15] C. Franks. Logical nihilism. In P. Rush, ed., *Metaphysics of Logic*, p. 109--127. Cambridge University Press, 2015.

[Fre79] G. Frege. *Begriffschrift*. Nebert, Halle, 1879. Translated in [Fre67].

[Fre67] G. Frege. Begriffschrift, a formal language, modelled upon that of arithmetic, for pure thought. In J. van Heijenoort, ed., *From Frege to Gödel: A Source Book in Mathematical Logic*, pp. 420–440. Harvard University Press, Cambridge MA, 1967. translated by S. Bauer-Mengelberg.

[Gla15] M. Glanzberg. Logical consequence and natural language. In O. Hjortalnd and C. Caret, eds., *Foundations of Logical Consequence*, pp. 71–120. Oxford University Press, 2015.

[Lak76] I. Lakatos. *Proofs and Refutations*. Cambridge University Press, 1976.

[Lau14] D. Lauer. Sprache als Spiel: *ergon* und *energeia*. In H. von Sass and S. Berg, eds., *Spielzüge: Zur Dialektik des Spiels und seinem metaphorischen Mehrwert*, pp. 224–263. Karl Alber, Freiburg, München, 2014.

[McG96] V. McGee. Logical operations. *J. Phil. Logic*, 25(6):567–580, 1996.

[Mor89] C. Mortensen. Anything is possible. *Erkenntnis*, 30(3):319–337, 1989.

[Per14] J. Peregrin. *Inferentialism: Why Rules Matter*. Palgrave Macmillan, London, 2014.

[Pri06] G. Priest. *Doubt Truth to Be a Liar*. Oxford University Press, 2006.

[PS17] J. Peregrin and V. Svoboda. *Reflective Equilibrium and the Principles of Logical Analysis*. Routledge, Milton Park, UK, 2017.

[PW18] G. Payette and N. Wyatt. Logical particularism. In Wyatt et al. [WPK18], pp. 277–299.

[Qui51] W. V. O. Quine. Two dogmas of empiricism. *Philosophical Review*, 60(1):20--43, 1951.

[Qui60] W. V. O. Quine. *Word and Object*. MIT Press, Cambridge, MA, 1960.

[Qui86] W. V. O. Quine. *Philosophy of Logic*. Harvard University Press, Cambridge MA, 1986.

[Rec03] F. Recanati. *Literal Meaning*. Cambridge University Press, 2003.

[Rus17] G. Russell. An introduction to logical nihilism. In H. Leitgeb, I. Niiniluoto, P. Seppälä, and E. Sober, eds., *Logic, methodology and philosophy of science. Proceedings of the 15th International Congress*, pp. 125–135. College Publications, London, 2017.

[Rus18] G. Russell. Logical nihilism: Could there be no logic? *Philosophical Issues*, 28(1):308–324, 2018.

[Sha14] S. Shapiro. *Varieties of Logic*. Oxford University Press, 2014.

[Str50] P. F. Strawson. On referring. *Mind*, 59(235):320–344, 1950.

[Wit53] L. Wittgenstein. *Philosophische Untersuchungen*. Blackwell, Oxford, 1953.

[WPK18] J. Wyatt, N. Pedersen, and N. Kellen, eds. *Pluralisms in Truth and Logic*. Palgrave Macmillan, London, 2018.

[Zach06] R. Zach. Hilbert's programme then and now. In D. Jacquette, ed., *Philosophy of Logic. Handbook of the Philosophy of Science, vol. 5*, chap. 11, pp. 411–447. Elsevier, 2006.

[Zar18] E. Zardini. Generalized Tarski's thesis hits substructure. In Wyatt et al. [WPK18], pp. 237–276.

2022 ACTA UNIVERSITATIS CAROLINAE PP. 27–44
**PHILOSOPHICA ET HISTORICA 1** / MISCELLANEA LOGICA

# ON LIFTING OF EMBEDDINGS BETWEEN TRANSITIVE MODELS OF SET THEORY

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**ABSTRACT**

Suppose $M$ and $N$ are transitive models of set theory, $\mathbb{P}$ is a forcing notion in $M$ and $G$ is $\mathbb{P}$-generic over $M$. An elementary embedding $j : (M, \in) \to (N, \in)$ *lifts* to $M[G]$ if there is $j^+ : (M[G], G, \in) \to (N[j^+(G)], j^+(G), \in)$ such that $j^+$ restricted to $M$ is equal to $j$. We survey some basic applications of the lifting method for both large cardinals and small cardinals (such as $\omega_2$, or successor cardinals in general). We focus on results and techniques which appeared after Cummings's handbook article [Cum10]: we for instance discuss a generalization of the surgery argument, liftings based on fusion, and compactness principles such as the tree property and stationary reflection at successor cardinals.
**Keywords:** lifting of embeddings; compactness principles; fusion arguments.

## 1 Introduction

Various results in set theory are derived by means of elementary embeddings between transitive models of set theory (or its fragments). An important part of these argument is the *lifting of elementary embeddings*. By this notion we mean the following: Suppose $M$ and $N$ are transitive models (sets or proper classes) of a sufficient fragment of ZFC and

$$j : (M, \in, \dots) \to (N, \in, \dots) \tag{1}$$

is an elementary embedding with critical point $\kappa \in M$. Assume further that $\mathbb{P} \in M$ is a forcing notion and $G$ is a $\mathbb{P}$-generic filter over $M$ (i.e. $G$ meets every maximal antichain of $\mathbb{P}$ which is an element of $M$). We say that $j$ *lifts to* $\mathbb{P}$ *(or* $G$*)* if there exists a $j(\mathbb{P})$-generic filter $H$ over $N$ such that $j$ extends to an elementary embedding

$$j^+ : (M[G], G, \in, \dots) \to (N[H], H, \in, \dots). \tag{2}$$

Notice that we include $G$ as an additional predicate in $M[G]$. See Theorem 2.1 for a sufficient and necessary condition for a lifting of $j$ to exist.

If $j : M \to N$ is an elementary embedding, we call $N$ the *target model* of $j$. Let us use $V$ to denote the current ambient universe.

Two main methods are used for lifting, in particular for finding the required generic filter $H$:

https://www.doi.org/10.14712/24647055.2025.2

(A) We find $H$ in $V[G]$ to retain the definability of $j^+$ in $V[G]$ (provided $j$ itself was definable in $V$ and $G$ is $\mathbb{P}$-generic over $V$). This is used for showing that $\kappa$ is preserved as a large cardinal.

In the simplest configuration, it is enough to construct $H$ by a counting argument which ensures that we meet all maximal antichains in $j(\mathbb{P})$ which are elements of the target model, while making sure that $H$ satisfies the necessary criterion for lifting, i.e. $j''G \subseteq H$ (see Theorem 2.1). The latter task is much easier if we can show that there exists $q \in j(\mathbb{P})$ which is below all elements in $j''G$. Such a $q$ is called a *master condition*: if $q$ is a master condition, then $q \in H$ implies $j''G \subseteq H$.

In other situations an ad hoc argument, or an argument specific for a given class of forcings, is often required for lifting: see Section 3 for a method based on modifying an existing generic filter, and Section 4 for the situation in which $j''G$ generates the required $H$.

(B) We force $H$ to exist in some further generic extension of $V[G]$. $\kappa$ may cease to be a large cardinal (depending on the nature of the generic extension), but it can still retain some desirable combinatorial properties (the tree property, stationary reflection, etc.).

There are two challenges in forcing $H$ to exist: First, we need to argue that $j(\mathbb{P})$ has reasonable properties over $V[G]$ over which we wish to force with it, in particular that it does not collapse cardinals we wish to preserve. This is not automatic even for very simple forcings $\mathbb{P}$—while $j(\mathbb{P})$ may have nice properties in the target model (by elementarity), its properties over $V[G]$ may be ill-behaved, or difficult to compute. Second, we need to argue that we can choose a $j(\mathbb{P})$-generic filter $H$ over $V[G]$ which contains $j''G$.

In case (B), if $j^+ : M[G] \to N[H]$ exists in some generic extension $V[G^*]$ which contains $V[G]$, we say that $j^+$ is a *generic elementary embedding*, meaning that it is added by $G^*$. The critical point of a generic elementary embedding is typically a small cardinal in $V[G]$, and may not be a cardinal in $V[G^*]$.

We will attempt to review the most important examples for both these methods, with focus on those which appeared only after the comprehensive and clearly written [Cum10] was published (but we will often refer to [Cum10] for context and definitions). The selection of the examples is subjective and is limited both by the length of the article and our preferences and knowledge. Here is a brief summary of the topics:

- Silver first showed how to obtain a measurable cardinal $\kappa$ with $2^\kappa = \kappa^{++}$ starting with a $\kappa^{++}$-supercompact cardinal $\kappa$ (see [Cum10, Section 12] for details). The argument uses a master condition for the lifting, making an essential use of the fact that if $G \subseteq \mathrm{Add}(\kappa, \kappa^{++})$ is a generic filter, then $j''G \subseteq j(\mathrm{Add}(\kappa, \kappa^{++}))$ is an element of the target model and therefore $\bigcup j''G$ is a legitimate condition in $j(\mathrm{Add}(\kappa, \kappa^{++}))$, which is used as a master condition. Magidor (see [Cum10, Section 13]) modified the argument by approximating the master condition by a diagonal construction, starting with just a $\kappa^+$-supercompact $\kappa$. Woodin showed that a much smaller large cardinal is sufficient (and is actually optimal) for obtaining a measurable cardinal which violates GCH: it suffices if there is an elementary embedding $j : V \to M$ with critical point $\kappa$ such that $j(\kappa) > \kappa^{++}$ and $M$ is closed under $\kappa$-sequences in $V$. Such a cardinal is called $\kappa^{++}$*-tall*. Tallness is an important weakening of a $\kappa + 2$*-strong* cardinal. Using the so called *surgery argument for the Cohen forcing*, such a $j$ can be lifted using a more complicated argument which is described in [Cum10, Section 25] and also slightly differently in [Cum92]. Woodin's argument follows case (B): first it is shown that a certain forcing of the form $i(\mathbb{P})$ behaves well over the current universe, an $i(\mathbb{P})$-generic filter $h$ is forced over the universe (where $i$ is a normal ultrapower embedding derived from the extender embedding $j$), a generic filter $H$ for $j(\mathbb{P})$ is constructed from $h$, and then $H$ is modified to $H^*$ which fits the criterion $j''G \subseteq H^*$. In Section 3 we briefly review Woodin's argument and follow up with a description of the technique from [CM14] which extends Woodin's argument to a more general setting of an Easton-like result for a cardinal $\kappa$ which is both $\lambda$-supercompact and $\mu$-tall for some regular $\lambda, \mu$ with $\kappa \leq \lambda < \lambda^{++} \leq \mu$. We also mention that the original Woodin's method can be used to obtain indestructibility of a degree of tallness or strongness under the Cohen forcing or the Mitchell forcing ([Ham09] and [Hon19]).

- The surgery method—powerful as it is—seems to be ill-suited for dealing with general iterations because it requires a manual modification of a generic filter to ensure $j''G \subseteq H$. It is harder to do this if conditions are composed of names. As it turns out, a $\lambda$-tall embedding with critical point $\kappa$ can be lifted more easily, provided the forcing notion we are lifting has certain "fusion-like" properties (for instance the generalized $\kappa$-Sacks forcing has them, but the $\kappa$-Cohen forcing does not).[1] This method originated in [FT08] and has been used since then to deal with more complex iterations. Unlike the surgery method, it does not use a manual modification of a filter; instead, it uses an observation that with a suitable $j$, for a generic filter $G \subseteq \mathbb{P}$, $j''G$ generates a generic filter for $j(\mathbb{P})$ (this is false for the $\kappa$-Cohen forcing but true for a version of the $\kappa$-Sacks forcing provided $j$ has certain properties). We briefly review this method in Section 4.

- While Sections 3 and 4 deal with cases (A) + (B) which preserve $\kappa$ as a large cardinal, Section 5 deals with case (B) in which the critical point is turned into a small successor cardinal. We will review how lifting is used to argue for the consistency of various compactness principles, such as the *tree property* and *stationary reflection*, at small successor cardinals (for instance $\omega_2$).

## 2 Preliminaries

We will follow the notation from [Cum10], where the reader finds all definitions which we are going to use here.

[1]To indicate on which cardinal $\kappa$ the current forcing lives, we often say $\kappa$-Sacks forcing, $\kappa$-Cohen forcing, etc.

In this section we briefly summarize some background information which we will use frequently.

## 2.1 Silver's lifting lemma

An observation due to Silver gives an if and only if condition for the existence of a lifting of an elementary embedding to a generic extension. We include this condition for completeness.

**Theorem 2.1** (Silver) *Let $j : M \to N$ be an elementary embedding between transitive models of* ZFC.[2] *Let $\mathbb{P} \in M$ be a forcing notion, let $G$ be $\mathbb{P}$-generic over $M$ and let $H$ be $j(\mathbb{P})$-generic over $N$. Then the following are equivalent:*

*(i)* $j''G \subseteq H$,

*(ii) There exists an elementary embedding $j^+ : M[G] \to N[H]$ such that $j^+(G) = H$ and $j^+ \restriction M = j$.*

It is easy to see that the lifted embedding $j^+$ has similar properties as $j$ (e.g. if $j$ is an extender embedding, so is $j^+$, and the supports are the same; see [Cum10, Section 9] for details).

## 2.2 Regular embeddings from elementary embeddings

Recall that if $\mathbb{P}$ and $\mathbb{Q}$ are forcing notions, then $i : \mathbb{P} \to \mathbb{Q}$ is called *regular embedding* if for all $p, q \in \mathbb{P}$, (i) $p \leq q \to i(p) \leq i(q)$, (ii) $p \perp q \leftrightarrow i(p) \perp i(q)$, and for every maximal antichain $A \subseteq \mathbb{P}$, $i''A$ is a maximal antichain in $\mathbb{Q}$.

The following is standard (see for instance [Cum10]).

**Fact 2.2** *Assume $\mathbb{P}, \mathbb{Q}$ are forcing notions, $G$ is a $\mathbb{P}$-generic filter, and $i : \mathbb{P} \to \mathbb{Q}$ is a regular embedding. Then $\mathbb{Q}$ is equivalent to $\mathbb{P} * \mathbb{Q}/\dot{G}$ where $\mathbb{Q}/\dot{G}$ is a $\mathbb{P}$-name for a forcing notion with conditions*

$$\{\, q \in \mathbb{Q}\,;\ q \text{ is compatible with } i''G \,\}, \tag{3}$$

*with the ordering inherited of $\mathbb{Q}$. We write $\mathbb{Q}/G$ for the interpretation of $\mathbb{Q}/\dot{G}$ in $V[G]$ and call $\mathbb{Q}/G$* the quotient of $\mathbb{Q}$ over $G$. *Sometimes, we also write $\mathbb{Q}/\mathbb{P}$ if a specific $G$ is not important.*

Notice that the quotient $\mathbb{Q}/\dot{G}$ is defined in $V$ and strictly speaking depends on $i$ (which will usually be given by the context). We find it useful to relativize this definition to transitive models of set theory other than $V$. Let $M$ be a transitive model of set theory and $\mathbb{P} \in M$ a forcing notion; we define $\mathrm{MaxAntichain}(\mathbb{P})^M$ to be the set of all maximal antichains of $\mathbb{P}$ which are elements of $M$.

**Definition 2.3** Let $M$ and $N$ be two transitive models of set theory and $\mathbb{P} \in M$ and $\mathbb{Q} \in N$ partial orders. We say that $i : \mathbb{P} \to \mathbb{Q}$ is an *$(M, N)$-regular embedding* if $i$ satisfies conditions (i) and (ii) from the definition of regular embedding and moreover $i''A \in \mathrm{MaxAntichain}(\mathbb{Q})^N$ for every $A \in \mathrm{MaxAntichain}(\mathbb{P})^M$.

[2] We assume everything happens in some ambient universe $V$ which contains $M, N, j$ as elements (if they are sets), or $M, N, j$ are definable in $V$ (if they are proper classes).

It is clear from the definition that if $i$ is an $(M, N)$-regular embedding, then whenever $H$ is $\mathbb{Q}$-generic over $N$, then $G = i^{-1}{}''H$ is $\mathbb{P}$-generic over $M$.

We will make use of the following fact:

**Fact 2.4** *Assume $j : M \to N$ is an elementary embedding with critical point $\lambda$ between a pair of transitive models of set theory and let $\mathbb{P} \in M$ be a partial order such that $M \models$ "$\mathbb{P}$ is $\lambda$-cc". Then the following hold:*

*(i) The restriction $j \restriction \mathbb{P} : \mathbb{P} \to j(\mathbb{P})$ is an $(M, N)$-regular embedding. In particular, if $H$ is $j(\mathbb{P})$-generic over $N$ and $G = j^{-1}{}''H$, then $j$ lifts to*

$$j : M[G] \to N[H]. \tag{4}$$

*(ii) Moreover, if*

$$j \restriction \mathbb{P} \in N \text{ and } \mathrm{MaxAntichain}(\mathbb{P})^N \subseteq \mathrm{MaxAntichain}(\mathbb{P})^M, \tag{5}$$

*then*

$$N \models j \restriction \mathbb{P} \text{ is a regular embedding from } \mathbb{P} \text{ into } j(\mathbb{P}) \text{ and } j(\mathbb{P}) \text{ is equivalent to } \mathbb{P} * j(\mathbb{P})/\dot{G}. \tag{6}$$

**Proof** (i) By elementarity, $j$ preserves the ordering relation and compatibility between $\mathbb{P}$ and $j(\mathbb{P})$. To argue for regularity, it suffices to show that if $A \in M$ is a maximal antichain in $M$, then $j''A \in N$ is a maximal antichain in $j(\mathbb{P})$. This follows immediately by elementarity and the fact $j''A = j(A)$, which holds since $M \models$ "$|A| < \lambda$", and $j$ is the identity below $\lambda$.

(ii) First notice that $j \restriction \mathbb{P} \in N$ implies that $\mathbb{P} = \mathrm{dom}(j \restriction \mathbb{P}) \in N$. To be able to carry out the quotient analysis from Fact 2.2 in $N$, it suffices to assume that $j \restriction \mathbb{P}$ is a regular embedding in $N$ which follows from the fact that it is an $(M, N)$-regular embedding and (5) holds. □

When $G$ is $\mathbb{P}$-generic over $N$ and item (ii) of Fact 2.4 applies, the definition of the quotient $j(\mathbb{P})/G$ is expressible in $N[G]$ and we can write:

$$j(\mathbb{P})/G = \{\, p^* \in j(\mathbb{P})\,;\ N[G] \models p^* \text{ is compatible with } j''G \,\}. \tag{7}$$

One could try to weaken the assumption (5) and ask just for $\mathbb{P}$ being an element of $N$, which would be easier to ensure in general. With the assumption that $\mathbb{P} \in N$, we could write $N[G]$; however it is not clear under which circumstances the quotient forcing $j(\mathbb{P})/G$ is an element of $N[G]$.

## 3 Surgery-type arguments

Recall the main part of Woodin's argument for lifting a $\kappa^{++}$-tall embedding $j : V \to M$ to the forcing iteration $\mathbb{P} = \mathbb{P}_\kappa * \mathrm{Add}(\kappa, \kappa^{++})$[3] where $\mathbb{P}_\kappa$ is the Easton-support iteration

[3]We identify conditions in $\mathrm{Add}(\kappa, \kappa^{++})$ with partial functions of size $< \kappa$ from $\kappa^{++} \times \kappa$ to 2.

of $\mathrm{Add}(\alpha, \alpha^{++})$ for inaccessible cardinals $\alpha < \kappa$.[4] Suppose $G * g$ is $\mathbb{P}$-generic over $V$, and there exists a generic filter $h_0$ over $V[G][g]$ for a certain $\kappa^{++}$-cc and $\kappa^{+}$-distributive forcing[5] $\mathbb{R}_0$ so that in $V[G][g][h_0]$:

- $j$ lifts to $j : V[G] \to M[G][g][\tilde{h}]$, for some filter $\tilde{h}$ for the tail of the iteration $j(\mathbb{P})$ defined on the interval$(\kappa, j(\kappa))$.
- $G * g * \tilde{h} * h_1$ is $j(\mathbb{P})$-generic over $M$, for some generic filter $h_1$ for the forcing $j(\mathrm{Add}(\kappa, \kappa^{++})^{V[G]}$.

It can be shown that if this configuration arises using the methods described in [Cum92] or [Cum10, Section 25], then $j\,''g \not\subseteq h_1$, and hence $j$ cannot be lifted. However, an additional argument—*the surgery*—is invoked which uses properties of the Cohen forcing to argue that in $V[G][g][h_0]$, there exists $h_2$ with the following properties:

- $G * g * \tilde{h} * h_2$ is $j(\mathbb{P})$-generic over $M$.
- $j\,''g \subseteq h_2$.

It follows that $j$ can be lifted to $j : V[G][g] \to M[G][g][\tilde{h}][h_2]$, and since $h_0$ was added by a $\kappa^+$-distributive forcing notion,[6] it is possible to lift $j$ further to

$$j : V[G][g][h_0] \to M[G][g][\tilde{h}][h_2][h_0^*], \text{ for some } h_0^*,$$

concluding that $\kappa$ is still measurable in $V[G][g][h_0]$.

The surgery argument itself uses some specific combinatorial properties of the Cohen forcing (see [Cum92, Subsection 6, Fact 2] for more details) and proceeds as follows: one can manually modify each $p \in h_1$ on the set $\mathrm{dom}(p) \cap j\,''(\kappa^{++} \times \kappa)$ to match $j\,''g$ (for any $p$, this set has size at most $\kappa$). Let us call this modified condition $p^*$. $h_2$ is the collection $\{\, p^* \,;\, p \in h_1 \,\}$. Once it is shown that $h_2$ is still $j(\mathrm{Add}(\kappa, \kappa^{++})^{V[G]})$-generic over $M[G][g][\tilde{h}]$, we are done because $j\,''g \subseteq h_2$ is now true by the construction.

Cody and Magidor [CM14] generalized the surgery method to a $\lambda$-supercompact cardinal $\kappa$ which is also $\mu$-tall for some regular $\mu$ with $\kappa \leq \lambda < \lambda^{++} \leq \mu$, performing surgery also on the "ghost coordinates". More precisely, they controlled by means of the Cohen forcing the continuum function on the interval $[\kappa, \lambda]$ while preserving the initial large-cardinal strength of $\kappa$. Woodin's argument does not apply directly because while $j(p) = j\,''p$ in Woodin's case, in the context of [CM14], if $p$ is a condition in $\mathrm{Add}(\delta, \alpha)$ for a regular $\delta$ in $(\kappa, \lambda]$, then in general $j(p)$ is no longer equal to its pointwise image $j\,''p$. The elements in $j(p) \setminus j\,''p$ are called the "ghost coordinates" (of the condition). This generalization is spelled out in [CM14, Lemma 4]. Note that their method is also limited to the Cohen forcing.

Incidentally, there are two presentations of Woodin's original construction which differ in the sequence of steps for obtaining $h_0$. The forcing $\mathbb{R}_0$ can be used either

[4] We assume for simplicity that $\kappa^{++} = (\kappa^{++})^M$. If not, it is possible to define the iteration using a function $f : \kappa \to \kappa$ which satisfies $j(f)(\kappa) = \kappa^{++}$; see [Git89] or [Ham09] for more details.

[5] In this particular case, $\mathbb{R}_0$ is equivalent to $\mathrm{Add}(\kappa^+, \kappa^{++})$ defined over a certain submodel of $V[G][g]$. See [Hon19, Section 3.1] for more details.

[6] Strictly speaking, this requires an extender representation of $j$ (see [Cum10, Proposition 15.1].

over $V[G][g]$ as described above (as is done in [Cum10, Section 25]), or forced beforehand as described in [Cum92] ([Cum92, Subsection 5, Fact 2] makes it possible). In the latter approach, the extra forcing can be tucked-in into a preliminary stage, allowing an indestructibility result for tall cardinals with respect to Cohen forcing of a fixed length (see [Ham09]) or strong cardinals of a given degree with respect to Cohen and Mitchell forcing up to a fixed length (see [Hon19]). However, one should bear in mind that in either approach, the size of $2^{\kappa^+}$ is increased non-trivially (proportionally to the length of the Cohen forcing which should preserve the largeness of $\kappa$), unlike the analogous Laver's indestructibility result for supercompact cardinals which retains GCH above $\kappa$ if it holds in $V$.

It is open whether a similar surgery argument is available for iterations. As we will see in the next section, lifting of iterations can be done using a different method which uses a fusion argument. However, the fusion argument yields only the least possible failure of GCH at a measurable $\kappa$: $2^\kappa = \kappa^{++}$. The reason is that the iteration has support of size $\le \kappa$. A surgery argument applied with a $\kappa^+$-cc iteration with $< \kappa$-support could possibly achieve $2^\kappa > \kappa^{++}$ in arguments such as [FH12].

We will review the fusion-based approach in the next section.

# 4 Fusion-type arguments

While the $\kappa$-Cohen forcing for a regular $\kappa \ge \omega$ is usually the easiest test example for many applications, it may not be the case for the lifting of elementary embeddings. In hindsight, Woodin's surgery argument overcomes obstacles which are specific for forcings with supports of size $< \kappa$ (and in particular for the $\kappa$-Cohen forcing). There are other forcings which add fresh subsets[7] of $\kappa$ and can be lifted without the need to provide extra generic filters which need to be modified later.

This was first observed by Friedman and Thompson in [FT08] for the $\kappa$-Sacks forcing. We will briefly review the method, but we will focus on the $\kappa$-Grigorieff forcing for more variety. Our exposition follows [HV16].

## 4.1 Grigorieff forcing at an inaccessible cardinal

Let $\kappa$ be an inaccessible cardinal. Unless we say otherwise, $I$ denotes a $\kappa$-complete proper ideal on $\kappa$.

**Definition 4.1** Let $\kappa$ be inaccessible[8] and let $I$ be a subset of $\mathscr{P}(\kappa)$. Let us define

$$P_I = \{ f \,\dot{:}\, \kappa \to 2 \,;\, \mathrm{dom}(f) \in I \},$$

where $f \,\dot{:}\, \kappa \to 2$ is a partial function from $\kappa$ to 2. Ordering is by the reverse inclusion: for $p, q$ in $P_I$, $p \le q \leftrightarrow p \supseteq q$.

[7] A set $x \subseteq \kappa$ is fresh in $V[G]$ if $x \cap \alpha \in V$ for all $\alpha < \kappa$ but $x \notin V$.

[8] Much of what follows also holds for a successor $\kappa = \mu^+$ provided $2^\mu = \mu^+$; we focus here on an inaccessible $\kappa$ for simplicity.

Notice that if we let $I$ be the ideal of bounded subsets of $\kappa$, we obtain the usual Cohen forcing.

**Definition 4.2** For $\alpha < \kappa$ write

$$p \leq_\alpha q \leftrightarrow p \leq q \ \& \ \mathrm{dom}(p) \cap (\alpha + 1) = \mathrm{dom}(q) \cap (\alpha + 1).$$

We say that $\langle\, p_\alpha \mid \alpha < \kappa \,\rangle$ is a *fusion sequence* if $p_{\alpha+1} \leq_\alpha p_\alpha$ for every $\alpha < \kappa$, and $p_\gamma = \bigcup_{\alpha<\gamma} p_\alpha$ for limit $\gamma$.

The following theorem is standard (see [HV16, Theorem 2.6]).

**Theorem 4.3** *Assume* GCH *and let $I$ be a $\kappa$-complete ideal extending the nonstationary ideal on $\kappa$ ($\kappa$ inaccessible). Then $P_I$ preserves cofinalities if and only if $I$ is a normal ideal.*

We will consider the following generalization of the definition of $\leq_\alpha$ and of the fusion construction. Let $I$ be a normal ideal on $\kappa$ and $S \in I^*$, where $I^*$ is the filter dual to $I$, i.e. $I^* = \{\, X \subseteq \kappa \,;\, \kappa \setminus X \in I \,\}$. We will assume that $S$ is composed of limit ordinals; this is without loss of generality because we can always shrink $S$ by intersecting it with the limit ordinals, and still stay in $I^*$. Let $P_I$ be the forcing defined above.

**Definition 4.4** Define the relation $\leq^S_\alpha$ as follows.
(i) if $\alpha$ is in $S$:

$$p \leq^S_\alpha q \leftrightarrow p \leq q \ \& \ \mathrm{dom}(p) \cap (\alpha + 1) = \mathrm{dom}(q) \cap (\alpha + 1)$$

(ii) if $\alpha$ is in $\kappa \setminus S$:

$$p \leq^S_\alpha q \leftrightarrow p \leq q \ \& \ \mathrm{dom}(p) \cap \alpha = \mathrm{dom}(q) \cap \alpha.$$

We say that $\langle\, p_\alpha \mid \alpha < \kappa \,\rangle$ is an $S$-fusion sequence if $p_{\alpha+1} \leq^S_\alpha p_\alpha$ for every $\alpha$ and $p_\gamma = \bigcup_{\alpha<\gamma} p_\alpha$ for limit $\gamma$.

Notice that $S = \kappa$ gives the original definition of $\leq_\alpha$ and fusion.

The following lemma is easy to check.

**Lemma 4.5** *Assume $I$ is a normal ideal on $\kappa$, and $S$ is a set in $I^*$ which contains only limit ordinals. Then $P_I$ is closed under limits of $S$-fusion sequences.*[9]

To prevent a possible misunderstanding, notice that to be a fusion sequence or an $S$-fusion sequence for $S \in I^*$ in $P_I$ are properties of certain sequences of conditions in the same underlying forcing notion $(P_I, \leq)$.

[9] That is, for every $S$-fusion sequence $\langle\, p_\alpha \mid \alpha < \kappa \,\rangle$, $p = \bigcup_{\alpha<\kappa} p_\alpha$ is a condition, with $p \leq^S_\alpha p_\alpha$ for every $\alpha < \kappa$.

### 4.2 Lifting of the Grigorieff forcing

Let us fix some notation first.

**Definition 4.6** Assume $\kappa$ is regular and $\mathrm{Club}(\kappa)$ is the closed unbounded filter on $\kappa$. Let $S$ be stationary. Define:

$$\mathrm{Club}(\kappa)[S] = \{\, X \subseteq \kappa\, ;\, \exists C \text{ closed unbounded in } \kappa \text{ and } X \supseteq S \cap C \,\}.$$

The following is routine.

**Lemma 4.7** *For every stationary $S$,* $\mathrm{Club}(\kappa)[S]$ *is a normal proper filter which contains $S$ and extends* $\mathrm{Club}(\kappa)$.

We will study the forcing $P_I$ with $I$ being the dual ideal of a normal proper filter of the form $\mathrm{Club}(\kappa)[S]$.

**Definition 4.8** Let $j : V \to M$ be an elementary embedding with critical point $\kappa$ from the universe into a transitive class $M$. We say that a normal ideal $I$ on $\kappa$ *lifts for* $(j, S)$ if

$$S \in I^* \text{ and } \kappa \in j(\kappa \setminus S).$$

**Example 4.9** The nonstationary ideal on $\kappa$ does not lift for any $(j, S)$ because $\kappa$ is an element of $j(C)$ for every closed unbounded subset $C$ of $\kappa$. For any regular $\mu < \kappa$, let $E^\mu_\kappa$ denote the set of all limit ordinals with cofinality $\mu$. If $I$ is dual to $\mathrm{Club}(\kappa)[E^\mu_\kappa]$, then $I$ lifts for $(j, E^\mu_\kappa)$ for any $j$.

**Definition 4.10** Let $P$ be a forcing notion and let $\kappa$ be a regular cardinal. Assume that every decreasing sequence of conditions in $P$ of length $\leq \kappa$ has an infimum in $P$ and let $X \subseteq P$ be given. Then

$$\begin{aligned}\mathrm{Cl}_{\leq\kappa}X = \{\, p \in P\, ;\, & \text{for some decreasing sequence } \langle\, p_\alpha \mid \alpha < \kappa \,\rangle \text{ with } p_\alpha \in X \\ & \text{for all } \alpha < \kappa, \text{ the infimum of } \langle\, p_\alpha \mid \alpha < \kappa \,\rangle \text{ is less or equal to } p \,\}\end{aligned} \quad (8)$$

is called the *$\kappa$-closure of $X$*.

It is easy to see that that if $X$ is a directed family (for every $x, y$ in $X$ there exists $z$ in $X$ such that $z \leq x \;\&\; z \leq y$) closed under limits of sequences of length less than $\kappa$, then $\mathrm{Cl}_{\leq\kappa}X$ is a filter in $P$.

The idea behind the lifting of the Grigorieff forcing is to argue that the $\kappa$-closure $\mathrm{Cl}_{\leq\kappa}(j\,''g)$, where $g$ is a generic for $\mathbb{P}_I$ and $I$ is a normal ideal on $\kappa$ which lifts for $(j, S)$ for some $S$, is already a generic filter for $j(P_I)$. This is in stark contrast with the $\kappa$-Cohen forcing $\mathrm{Add}(\kappa, 1)$: If $g$ is $\mathrm{Add}(\kappa, 1)$-generic, then the $\kappa$-closure of $j\,''g$ is equal just to $g \cup \{\bigcup g\}$ which yields a function with domain $\kappa$ while every $j(\mathrm{Add}(\kappa, 1))$-generic must yield a function with domain $j(\kappa)$. The reason is that for every $p \in \mathrm{Add}(\kappa, 1)$, $j(p) = p$ because $|\mathrm{dom}(p)| < \kappa$. Allowing conditions with $|\mathrm{dom}(p)| = \kappa$ as in $P_I$ overcomes this limitation.

Let us show how the argument works for the simple case of a normal measure ultrapower. Assume $\mathsf{GCH}$ and let $j : V \to M$ be an ultrapower embedding with critical point $\kappa$. In particular $M = \{ j(f)(\kappa) ; f : \kappa \to V \}$. Consider a forcing of the form $\mathbb{P} * \dot{P}_I$, where $\mathbb{P}$ is a reverse Easton iteration with $\mathbb{P} \subseteq V_\kappa$ and $\dot{P}_I$ is a $\mathbb{P}$-name for the Grigorieff forcing, where $I$ is a normal ideal which lifts for $(j, S)$ for some $S$. Think of $\mathbb{P}$ as the standard preparation for $P_I$. Let $G * g$ be $\mathbb{P} * \dot{P}_I$-generic over $V$ and assume we can lift $j$ partially to $j : V[G] \to M^* = M[G][g][H]$ for some $H \in V[G][g]$. It will hold that

$$M^* = \{ j(f)(\kappa) ; f \in V[G] : \kappa \to V[G] \}. \tag{9}$$

**Lemma 4.11** $\mathrm{Cl}_{\leq\kappa}(j''g)$ *is a* $j(P_I)$*-generic filter over* $M^*$.

**Proof** Let us denote $h = \mathrm{Cl}_{\leq\kappa}(j''g)$. It is clear that $h$ is a filter and is well-defined because by standard arguments, $M^*$ is closed under $\kappa$-sequences in $V[G * g]$, and $j(P_I)$ is $\kappa^+$-closed in $M^*$.

By (9), every dense open set in $j(P_I)$ has the form $j(f)(\kappa)$ for an $f : \kappa \to H(\kappa^+)^{V[G]}$ in $V[G]$. Moreover, we can assume that $\langle f(\alpha) \mid \alpha < \kappa \rangle$ is in $V[G]$ a sequence of dense open sets in $P_I$ for every such $f$. Let us fix a dense open set $D = j(f)(\kappa)$. It suffices to show that for any $p$, there is a condition $p^* \leq p$ which satisfies the following items:

(i) $p^*$ is a limit of an $S$-fusion sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$ such that $\alpha \in \mathrm{dom}(p_\alpha)$ for every $\alpha \in \kappa \setminus S$.

(ii) For every $\alpha < \kappa$, whenever $d$ is a condition with $\mathrm{dom}(d) = \alpha + 1$ which extends the (partial) condition $p^* \restriction (\alpha + 1)$, then $d \cup (p^* \restriction [\alpha + 1, \kappa))$ is in the dense open set $f(\alpha)$.

It is easy to construct such a sequence using the fusion properties of $P_I$.

We argue as follows to show that (i) and (ii) are sufficient: by density, there is some such $p^*$ in $g$. By (i) and (ii), elementarity and by $I$ lifting for $(j, S)$, $p^{**} = \bigcup g \cup j(p^*)$ is a condition whose domain includes $\kappa + 1$ and is therefore an element of $j(f)(\kappa) = D$. It is also easy to see that $p^{**}$ is in $h$, and we are done. □

**Remark 4.12** The lifting of $j$ described in the previous paragraph is not very interesting because $j$ is just a normal measure embedding and $2^\kappa = \kappa^+$. However, the argument naturally generalizes to a $(\kappa, \lambda)$-extender embeddings $j$, with $\kappa^+ < \lambda$ regular. Let $h : \kappa \to \kappa$ be chosen so that $j(h)(\kappa) \geq \lambda$. We can assume that every dense open set in $j(P_I)$ is of the form $j(f)(\delta)$ for some $f : \kappa \to H(\kappa^+)^{V[G]}$ and $\delta < \lambda$. Let us fix an arbitrary dense open set $D = j(f)(\delta)$. It suffices to modify the properties of the $S$-fusion sequence $\langle p_\alpha | \alpha < \kappa \rangle$ mentioned above so that the condition (ii) above requires that $d \cup (p^* \restriction [\alpha + 1, \kappa))$ should be in $\bigcap_{\beta < h(\alpha)} f(\beta)$. Then $\bigcup g \cup j(p^*)$ meets every dense open set indexed below $j(h)(\kappa)$, which means it meets $D$. With lifting available for extender embeddings, one can use a $\leq \kappa$-supported product[10] of the forcings $P_I$ to reprove—without a surgery argument—Woodin's original result (see [HV16] for more details).

[10] Notice that in dealing with the product of the $\kappa$-Grigorieff forcing, we need to deal with "ghost coordinates", similarly as we discussed in Section 3. However, there is a difference: the ghost coordinates for the Cohen forcing appear only if we force over a regular cardinal larger than $\kappa$, while with the $\kappa$-Grigorieff forcing this phenomenon appears already at stage $\kappa$—the reason is of course that the conditions in the $\kappa$-Grigorieff forcing have size $\leq \kappa$, and the support of the product has also size $\leq \kappa$.

### 4.3 Generalizations and applications

Let us discuss some background information and applications of the method discussed in Section 4.2.

(1) The lifting via fusion was introduced in [FT08] using the $\kappa$-Sacks forcing (see [Kan80] for more details about the $\kappa$-Sacks forcing): the forcing is composed of $\kappa$-perfect trees viewed as subsets of $2^{<\kappa}$ which have continuous splitting: whenever $\langle x_\alpha \mid \alpha < \delta \rangle$ is a strictly extending sequence of nodes in a tree $p$ with $\delta < \kappa$ a limit ordinal, then if the splitting nodes are unbounded in $x = \bigcup\{ x_\alpha \ ; \ \alpha < \delta \}$, then $x$ is a splitting node in $p$. This definition has the effect that $j\,''g$ (in the notation of the previous section) does not generate a generic filter because every tree $j(p)$, with $p \in g$, splits at level $\kappa$ (this feature was dubbed the "tuning fork argument" in [FT08]). While $j\,''g$ does not generate a generic filter, it "almost" generates it: once we choose for every $p \in g$ whether we go to the left or to the right on the level $\kappa$ in $j(p)$ (consistently for all $p$), then we do get a generic filter. On the other hand, we may slightly modify the definition of the forcing to ensure that $j\,''g$ generates a generic filter: it suffices to modify the definition of the forcing to require the continuous splitting only for $\delta$'s of a certain cofinality (such as $\delta$ in $E^\mu_\kappa$ in Example 4.9).

The control of cofinality of $\delta$ is also used in our treatment of the $\kappa$-Grigorieff forcing: the stationary set $S$ in Definition 4.4 controls which ordinals can be added to the domains of conditions in a fusion sequence and which may not be added and consequently controls whether $j\,''g$ generates a generic filter.

This flexibility of controlling the number of possible generic filters, and consequently the number of liftings,—exactly one for the $\kappa$-Grigorieff forcing (e.g. with an ideal containing the complement of $E^\mu_\kappa$) and exactly two for the $\kappa$-Sacks forcing in [FT08]—was exploited in a paper by Friedman and Magidor [FM09]. They generalized the definition of the $\kappa$-perfect tree and controlled the number of normal measures at $\kappa$ in the final model by prescribing the size of the set of continuations of a splitting node.

(2) An important advantage of the lifting with fusion is the ability to handle iterations. In [FH12], a model is constructed in which $2^{\aleph_\omega} = \aleph_{\omega+2}$, $\aleph_\omega$ is strong limit, and there is a well-ordering of the subsets of $\aleph_\omega$ lightface definable in $H(\aleph_{\omega+1})$. The argument starts with a $(\kappa+2)$-strong $\kappa$ in an extender model $L[\vec{E}]$. Over this model, a cofinality-preserving Easton-supported iteration $\mathbb{P} = \lim\langle (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha) \,|\, \alpha \le \kappa \rangle$ is defined where for each inaccessible $\alpha \le \kappa$, $\dot{\mathbb{Q}}_\alpha$ adds (among other things) $\alpha^{++}$-many new subsets to $\alpha$ using a version of the $\alpha$-Sacks forcing. The future well-ordering of the subsets of $\alpha$ is coded by means of selective kills of certain stationary subsets of $\alpha^+$; this information is in turn localized by an $\alpha^+$-distributive forcing which ensures the lightface definability in $H(\alpha^+)$. In the context of this survey it is important that the iteration $\dot{\mathbb{Q}}_\kappa$ may be lifted using a fusion-type argument along the lines of Section 4.2, without any surgery. Since $\kappa$ remains measurable after forcing with $\mathbb{P}$, the proof concludes by using a version of the Prikry forcing with collapses to turn $\kappa$ to $\aleph_\omega$ while preserving the definability of the well-order.

At the moment, results like this seem to be out of reach for any method based on surgery.[11] For instance, it is an open problem how to achieve a definability result such as [FH12] with a gap larger than 2, for instance to have $2^\kappa > \kappa^{++}$, $\kappa$ measurable, and a well-ordering of subsets of $\kappa$ (lightface) definable in $H(\kappa^+)$.

(3) We may attempt to characterize the class of forcings which add fresh subsets to a measurable cardinal $\kappa$ and which can be lifted using an argument based on fusion. Interestingly, this characterization (or rather the resulting class of forcings) is very similar to a class of forcings with conditions of size $\kappa$ for which a reasonable notion of $\kappa$-properness may be formulated. See for instance [FHZ13], [HS20], and [RS19] for more details.

(4) In [FH08], the $\kappa$-Sacks forcing was used to prove a version of Easton's theorem for the continuum function while preserving certain large cardinals.

# 5 Generic embeddings

In this section, we discuss case (B) from Section 1, with focus on the tree property and stationary reflection. Recall the following definitions:

**Definition 5.1** Let $\lambda$ be a regular cardinal. We say that the *tree property* holds at $\lambda$, and we write $\mathsf{TP}(\lambda)$, if every $\lambda$-tree has a cofinal branch.

**Definition 5.2** Let $\lambda$ be a cardinal of the form $\lambda = \nu^+$ for some regular cardinal $\nu$. We say that the *stationary reflection* holds at $\lambda$, and write $\mathsf{SR}(\lambda)$, if every stationary subset $S \subseteq \lambda \cap \mathrm{cof}(< \nu)$ reflects at a point of cofinality $\nu$; i.e. there is $\alpha < \lambda$ of cofinality $\nu$ such that $\alpha \cap S$ is stationary in $\alpha$.

More information about these properties can be found in [Cum05] and [Jech03].

## 5.1 The tree property

Let us start with a quick review of a typical argument which uses lifting of an embedding to obtain a large cardinal property at a small cardinal. We sketch the argument that if there is a weakly compact cardinal $\lambda$, then there is a generic extension where the tree property holds at $\omega_2$.

Recall the following definition which is implicit in [Mit72] and the present form is taken from [Abr83].

**Definition 5.3** Suppose $\omega \leq \kappa \leq \lambda$ are regular cardinals and $\lambda$ is inaccessible. Conditions in the Mitchell forcing, $\mathbb{M}(\kappa, \lambda)$, are pairs $(p^0, p^1)$ such that $p^0 \in \mathrm{Add}(\kappa, \lambda)$ and $p^1$ is a function with domain $\mathrm{dom}(p^1) \subseteq \lambda$ of size at most $\kappa$. For $\alpha$ in the domain of $p^1$, $p^1(\alpha)$ is an $\mathrm{Add}(\kappa, \alpha)$-name and

$$1_{\mathrm{Add}(\kappa,\alpha)} \Vdash p^1(\alpha) \in \mathrm{Add}(\kappa^+, 1)^{V[\mathrm{Add}(\kappa,\alpha)]}. \quad (10)$$

[11] Note that supercompact cardinals—which lift more easily—will not help here since there are no canonical inner models for supercompact cardinals such as $L[\vec{E}]$ for the coding to work properly in the non-$\mathsf{GCH}$ context.

The ordering is defined as follows: $(p^0, p^1) \le (q^0, q^1)$ iff $p^0 \le q^0$ in $\mathrm{Add}(\kappa, \lambda)$ and the domain of $p^1$ extends the domain of $q^1$, and for every $\alpha \in \mathrm{dom}(q^1)$,

$$p^0 \restriction \alpha \Vdash_{\mathrm{Add}(\kappa,\alpha)} p^1(\alpha) \le q^1(\alpha), \tag{11}$$

where $p^0 \restriction \alpha$ is the restriction of $p^0$ to $\mathrm{Add}(\kappa, \alpha)$.

If $\kappa, \lambda$ are understood from the context, we write just $\mathbb{M}$. For $\alpha < \lambda$, let $\mathbb{M}_\alpha$ denote the natural truncation of $\mathbb{M}$ to $\alpha$ (we write $(p^0, p^1) \restriction \alpha$ for the restriction of $(p^0, p^1)$ to $\mathbb{M}_\alpha$).

Using the Abraham's analysis (see [Abr83]), there is a projection onto $\mathbb{M}$ from the product $R^0 \times R^1$ where $R^0 = \mathrm{Add}(\kappa, \lambda)$ is $\kappa^+$-Knaster (under the assumption $\kappa^{<\kappa} = \kappa$) and $R^1$ is $\kappa^+$-closed (the "term" forcing). This analysis also holds for the natural quotients of $\mathbb{M}$ and $R^0$ and $R^1$: in particular,

$$\text{if } \alpha < \lambda \text{ is inaccessible, then there is a projection onto } \mathbb{M}/\mathbb{M}_\alpha \text{ from } R^0_\alpha \times R^1_\alpha, \tag{12}$$

where, under the appropriate assumptions, the forcing $R^0_\alpha$ is $\kappa^+$-Knaster (in fact, it is equivalent to $\mathrm{Add}(\kappa, \lambda)$) and $R^1_\alpha$ is $\kappa^+$-closed in $V[\mathbb{M}/\mathbb{M}_\alpha]$.

Finally recall that if $\lambda$ is weakly compact, then this fact is witnessed by the existence of elementary embeddings with critical point $\lambda$ between transitive models of $\mathsf{ZFC}^-$ of size $\lambda$ which are closed under $< \lambda$-sequences, and equivalently, $\lambda$ satisfies the $\Pi^1_1$-reflection (for more details, see [Cum10] or [Kan03]).

**Theorem 5.4** *Suppose $\lambda$ is weakly compact. Then $\mathbb{M} = \mathbb{M}(\omega, \lambda)$ forces $\lambda = \aleph_2$ and $\mathsf{TP}(\omega_2)$.*

**Proof** Easton's lemma shows that $R^0 \times R^1$, and hence $\mathbb{M}$, preserves $\omega_1$ and by design $\mathbb{M}$ turns $\lambda$ to $\omega_2$. Let us now argue for the tree property. Suppose for contradiction there is an $\omega_2$-Aronszajn tree $T$ in $V[G]$, where $G$ is $\mathbb{M}$-generic. It is illustrative to look at $\mathbb{M} = \mathbb{M}(\omega, \lambda)$ as a mixed support iteration which at many inaccessible stages $\alpha < \lambda$ deals with the restriction of $T \restriction \alpha$: for many such $\alpha$, $T \restriction \alpha$ is an element of $V[G_\alpha]$, it is an $\alpha = \omega_2^{V[G_\alpha]}$-tree, and morevover by the $\Pi^1_1$-reflection of $\lambda$ in the ground model, $T \restriction \alpha$ is an $\alpha$-Aronszajn tree in $V[G_\alpha]$. However, there must some node of height $\alpha$ in the whole tree $T$ which means that the forcing $\mathbb{M}/G_\alpha$ must add a cofinal branch to $T \restriction \alpha$. This is a contradiction since the product $R^0_\alpha \times R^1_\alpha$ cannot add such a branch on account of the so called "branch lemmas",[12] and hence neither can $\mathbb{M}/G_\alpha$.

The elementary submodel argument (which is behind the argument in the previous paragraph) is more often formulated in the language of elementary expansions and embeddings so it fits into our survey of lifting methods. Since $\lambda$ is weakly compact, we can choose a transitive model $M$ of size $\lambda$ closed under $< \lambda$-sequences which contains all

[12] Variants of the the following two: (1) If $2^\omega = \mu$ for some regular $\mu$ (or singular with uncountable cofinality), then no $\sigma$-closed forcing can add a cofinal branch to a $\mu$-tree. (2) If $\mathbb{P}$ is a forcing notion such that $\mathbb{P} \times \mathbb{P}$ is ccc, then $\mathbb{P}$ does not add cofinal branches to trees whose height has cofinality $\omega_1$. Useful generalizations appeared for instance in [Dev78, Tod81, JS90].

necessary parameters, in particular $\mathbb{M}$ and an $\mathbb{M}$-name $\dot{T}$ for an $\omega_2$-Aronszajn tree, and for which there is an elementary embedding $j : M \to N$ with critical point $\lambda$ into a transitive model $N$ of size $\lambda$ which is closed under $< \lambda$-sequences. Let $H$ be $j(\mathbb{M})$-generic over $V$. $N[H]$ is a generic extension by $j(\mathbb{M})$, and since $\mathbb{M}$ is $\lambda$-cc, by Fact 2.4 we know that $j^{-1}{}''H = G$ is $\mathbb{M}$-generic over $M$ and $j$ lifts to

$$j : M[G] \to N[H].$$

Now the argument finishes as in the first paragraph when we apply it in $N[H]$ and consider the restriction $j(T) \restriction \lambda = T$. □

Notice that in the previous proof, $j$ restricted to $\mathbb{M}$ is the identity, so we in fact have $H = G * H^*$ where $H^*$ is a generic filter over $N[G]$ for the tail iteration $j(\mathbb{M})$ from $\lambda$ to $j(\lambda)$. But this simple analysis of $H$ does not suffice in more complex constructions. Let us consider the following example:

**Example** This example is a simplified version of [FHS20] and shows how to get the tree property at the double successor of a singular strong limit cardinal with cofinality $\omega$. Suppose $\mathbb{M}(\kappa, \lambda)$ forces that $\kappa$ is measurable and let $\mathrm{Prk}(\dot{U})$ be the vanilla Prikry forcing which uses a normal measure $\dot{U}$ in $V[\mathbb{M}(\kappa, \lambda)]$ to add a cofinal sequence of type $\omega$ to $\kappa$ without collapsing any cardinals. Then in analogy with Theorem 5.4, we consider $j : M \to N$ and forcing notions $\mathbb{P} = \mathbb{M}(\kappa, \lambda) * \mathrm{Prk}(\dot{U}) \in M$ and $j(\mathbb{P}) \in N$. Since the quotient $j(\mathbb{P})/\mathbb{P}$ is no longer a "naturally" defined tail iteration of $j(\mathbb{P})$ and $j \restriction \mathbb{P}$ is not the identity, the lifting of $j : M \to N$ now proceeds as follows: Start by having $H$ which is $j(\mathbb{P})$-generic over $V$; then $G = j^{-1}{}''H$ is $\mathbb{P}$-generic over $M$ and $j$ lifts to $j : M[G] \to N[H]$. With some additional assumptions, as in Fact 2.4(ii), the quotient forcing $j(\mathbb{P})/G$ is an element of $N[G]$ and the argument finishes by showing that $j(\mathbb{P})/G$ does not add cofinal branches to $\lambda$-trees over $N[G]$. The problem now is that the quotient is not a natural forcing, but a very complex one, so the easy branch lemmas do not apply here. This obstacle can be overcome by a "hands-on" argument as in [FHS20], where a Prikry forcing with collapses is considered, or by an appeal to indestructibility of the tree property if we in addition assume that the normal measure $\dot{U}$ for the definition of $\mathrm{Prk}(\dot{U})$ lives already in $V[\mathrm{Add}(\kappa, \lambda)]$ (see Section 5.3 for more details).

**Remark 5.5** For the tree property at $\omega_2$, there is a more robust way of "sealing-off" an $\alpha$-Aronszajn tree $T \restriction \alpha$ mentioned above. Using the ideas from the argument that $\mathsf{PFA}$ implies $\mathsf{TP}(\omega_2)$, one can define a countable support iteration which at many inaccessible cardinals $\alpha < \lambda$ with $2^\omega = \alpha = (\omega_2)^{V[\mathbb{P}_\alpha]}$ first collapses $\alpha$ to $\omega_1$ and then specialize $T \restriction \alpha$ by a ccc forcing. After specialization, the tail iteration of $\mathbb{P}$ after stage $\alpha$ cannot add a cofinal branch to $T \restriction \alpha$ unless $\omega_1$ is collapsed, so there is no need to use any "branch lemmas". This makes it possible to consider complex countable support iterations $\mathbb{P}$ which preserve $\omega_1$ so that both $\mathsf{TP}(\omega_2)$ and some other properties hold in $V[\mathbb{P}]$ (such as $\mathsf{MA}$). It is not known whether such a robust method of "sealing-off" an $\alpha$-Aronszajn tree works also for regular cardinals greater than $\omega_2$.

### 5.2 Stationary reflection

The lifting argument for stationary reflection follows the same pattern as we discussed in Theorem 5.4:

**Theorem 5.6** *Suppose $\lambda$ is weakly compact. Then $\mathbb{M} = \mathbb{M}(\omega, \lambda)$ forces $\lambda = \aleph_2$ and* $\mathsf{SR}(\omega_2)$.

**Proof** In analogy with the proof of Theorem 5.4, suppose for contradiction there is a non-reflecting stationary set $S$ in $V[G]$ which concentrates on ordinals with cofinality $\omega$. At many inaccessible $\alpha < \lambda$, $S \cap \alpha$ is in $V[G_\alpha]$ a stationary set in $\alpha = (\omega_2)^{V[G_\alpha]}$ concentrating on ordinals with cofinality $\omega$ (by $\Pi^1_1$-reflection of $\lambda$ in the ground model). However, since $S$ is supposed to be non-reflecting, $S \cap \alpha$ cannot be stationary in $V[G]$: the contradiction is achieved by showing that the tail iteration $\mathbb{M}/G_\alpha$ does not destroy the stationarity of $S \cap \alpha$. Instead of "branch lemmas" we use "stationarity preserving lemmas",[13] which we apply to $R^0_\alpha \times R^1_\alpha$. □

Let us note that $\mathsf{SR}(\omega_2)$ does not imply $2^\omega > \omega_1$ so there is a greater variety of forcing notions to obtain stationary reflection. Also, it is known that a Mahlo cardinal is enough to get $\mathsf{SR}(\omega_2)$, see [HS85]. But a weakly compact cardinal is necessary for stronger forms of stationary reflection (see [Mag82]). Finally note that $\mathsf{TP}$ and $\mathsf{SR}$ do not imply one another, see [CFM+18].

**Remark 5.7** In analogy with the tree property at $\omega_2$—but modified to deal with $S$-proper forcings which unlike proper forcings (see [She16] for more details) may destroy stationary sets—it is possible to "seal-off" $S \cap \alpha$ by shooting a club through $S \cap \alpha$ by means of an $\omega_1$-distributive forcing so that it remains stationary in any extension which preserves $\omega_1$.

### 5.3 Indestructibility

Instead of "branch lemmas" and "stationarity preserving lemmas" with the—often technically difficult—analysis of quotients, as in $j(\mathbb{M}(\kappa, \lambda) * \mathrm{Prk}(\dot{U}))/\mathbb{M}(\kappa, \lambda) * \mathrm{Prk}(\dot{U})$ mentioned in the example in Section 5.1, one can attempt to formulate a more general preservation theorem. With such preservation, or indestructibility, theorems one can argue more easily for instance that $\mathbb{M}(\kappa, \lambda) * \mathrm{Prk}(\dot{U})$ forces $\mathsf{TP}(\lambda)$ and $\mathsf{SR}(\lambda)$ because $\mathbb{M}(\kappa, \lambda)$ does, and the relevant properties are preserved by $\mathrm{Prk}(\dot{U})$. The lifting argument is thus limited to $\mathbb{M}(\kappa, \lambda)$.

Stationary reflection is easier to handle because stationary sets are subsets of ordinals, while trees are binary relations on ordinals. In [HS22], the following is showed:

**Theorem 5.8** *Suppose $\lambda$ is a regular cardinal,* $\mathsf{SR}(\lambda^+)$ *holds and $\mathbb{Q}$ is $\lambda$-cc. Then* $\mathsf{SR}(\lambda^+)$ *holds in $V[\mathbb{Q}]$.*

[13] Most importantly, if $\kappa$ is regular, than no $\kappa$-cc or $\kappa$-closed forcing can destroy the stationarity of a subset of $\kappa$. In our case we need a variant which says that a countably closed forcing cannot destroy stationarity of sets concentrating on ordinals with countable cofinality.

**Proof** Suppose for contradiction there are $p_0 \in \mathbb{Q}$ and $\dot{S}$ such that $p_0$ forces that $\dot{S}$ is a non-reflecting stationary subset of $\lambda^+ \cap \mathrm{cof}(<\lambda)$. Set

$$U_{p_0} = \{ \gamma \in \lambda^+ \cap \mathrm{cof}(<\lambda) \,;\, \exists p \le p_0 \; p \Vdash \gamma \in \dot{S} \}. \tag{13}$$

$U_{p_0}$ is a stationary set: for every club $C \subseteq \lambda^+$, $p_0$ forces $C \cap \dot{S} \neq \emptyset$, and because $p_0$ also forces $\dot{S} \subseteq U_{p_0}$, it forces $C \cap U_{p_0} \neq \emptyset$, which is equivalent to $C \cap U_{p_0}$ being non-empty in $V$. By $\mathsf{SR}(\lambda^+)$ there is some $\alpha < \lambda^+$ of cofinality $\lambda$ such that

$$U_{p_0} \cap \alpha \text{ is stationary.} \tag{14}$$

By our assumption

$$p_0 \Vdash \dot{S} \cap \alpha \text{ is non-stationary.} \tag{15}$$

We will argue that (14) and (15) are contradictory, which will finish the proof.

First recall that by the $\lambda$-cc of $\mathbb{Q}$, every club subset of an ordinal $\alpha$ of cofinality $\lambda$ in $V[\mathbb{Q}]$ contains a club in the ground model. It follows by (15) that there is a maximal antichain $A$ below $p_0$ such that for every $p \in A$ there is some club $D$ in $\alpha$ in the ground model with $p \Vdash \dot{S} \cap D = \emptyset$. Let us fix for each $p \in A$ some $D_p$ such that $p \Vdash \dot{S} \cap D_p = \emptyset$.

Set

$$C = \bigcap \{ D_p \,;\, p \in A \}. \tag{16}$$

$C$ is a club subset of $\alpha$ because $A$ has size $<\lambda$ and $\alpha$ has cofinality $\lambda$. It holds

$$p_0 \Vdash \dot{S} \cap C = \emptyset \tag{17}$$

because conditions forcing $\dot{S} \cap C = \emptyset$ are dense below $p_0$: for every $q \le p_0$ there is some $p \in A$ which is compatible with $q$, and any $r \le p, q$ forces $\dot{S} \cap D_p = \emptyset$. Since $C \subseteq D_p$, this implies $r \le q$ forces $\dot{S} \cap C = \emptyset$.

However, by (14) there must be $\gamma \in C \cap U_{p_0} \cap \alpha$, and therefore some $p \le p_0$ such that $p \Vdash \gamma \in \dot{S} \cap C$. This contradicts (17). $\square$

In particular $\mathbb{M}(\kappa, \lambda) * \mathrm{Prk}(\dot{U})$ forces $\mathsf{SR}(\lambda)$. By a more technical argument, one can show (see [HS20]):

**Theorem 5.9** *Assume $\omega \le \kappa < \lambda$ are cardinals, $\kappa^{<\kappa} = \kappa$ and $\lambda$ is weakly compact. Let $\mathbb{M}$ be the standard Mitchell forcing $\mathbb{M}(\kappa, \lambda)$. Suppose $\mathbb{Q} \in V[\mathrm{Add}(\kappa, \lambda)]$ is $\kappa^+$-cc in $V[\mathrm{Add}(\kappa, \lambda)]$ (equivalently $\kappa^+$-cc in $V[\mathbb{M}]$), then*

$$V[\mathbb{M} * \dot{\mathbb{Q}}] \models \mathsf{TP}(\kappa^{++}).$$

*In other words, the tree property at $\kappa^{++}$ is indestructible under any $\kappa^+$-cc forcing which lives in $V[\mathrm{Add}(\kappa, \lambda)]$.*

This theorem suffices to argue that $\mathbb{M}(\kappa, \lambda) * \mathrm{Prk}(\dot{U})$, with $\dot{U}$ being a measure in $V[\mathrm{Add}(\kappa, \lambda)]$, forces $\mathsf{TP}(\lambda)$. In fact, the same theorem suffices also for the Magidor forcing in place of Prk to obtain a singular $\kappa$ with uncountable cofinality. It is open, however, whether it applies to Prikry forcing with collapses.

For completeness, let us mention that [GK09] gives an indestructibility argument for another compactness principle, the failure of the approachability property (see [Cum05] for a definition): the failure of the approachability property at $\kappa^{++}$ is preserved by $\kappa$-centered forcings.

**Acknowledgement** The author was supported by GAČR grant *The role of set theory in modern mathematics* (24-12141S).

## References

[Abr83] U. Abraham. Aronszajn trees on $\aleph_2$ and $\aleph_3$. *Ann. Pure Appl. Logic*, 24(3):213–230, 1983.

[CFM+18] J. Cummings, S.-D. Friedman, M. Magidor, A. Rinot, and D. Sinapova. The eightfold way. *J. Symb. Logic*, 83(1):349–371, 2018.

[CM14] B. Cody and M. Magidor. On supercompactness and the continuum function. *Ann. Pure Appl. Logic*, 165(2):620–630, 2014.

[Cum92] J. Cummings. A model in which GCH holds at successors but fails at limits. *Trans. Amer. Math. Soc.*, 329(1):1–39, 1992.

[Cum05] J. Cummings. Notes on singular cardinal combinatorics. *Notre Dame J. Formal Logic*, 46(3):251–282, 2005.

[Cum10] J. Cummings. Iterated forcing and elementary embeddings. In M. Foreman and A. Kanamori, eds., *Handbook of Set Theory*, vol. 2, chap. 12, pp. 775–883. Springer, 2010.

[Dev78] K. Devlin. $\aleph_1$-trees. *Ann. Math. Logic*, 13:267–330, 1978.

[FH08] S.-D. Friedman and R. Honzík. Easton's theorem and large cardinals. *Ann. Pure Appl. Logic*, 154(3):191–208, 2008.

[FH12] S.-D. Friedman and R. Honzík. A definable failure of the Singular Cardinal Hypothesis. *Israel J. Math.*, 192(2):719–762, 2012.

[FHS20] S.-D. Friedman, R. Honzík, and Š. Stejskalová. The tree property at $\aleph_{\omega+2}$ with a finite gap. *Fundamenta Mathematicae*, 251:219–244, 2020.

[FHZ13] S.-D. Friedman, R. Honzík, and L. Zdomskyy. Fusion and large cardinal preservation. *Ann. Pure Appl. Logic*, 164:1247–1273, 2013.

[FM09] S.-D. Friedman and M. Magidor. The number of normal measures. *J. Symb. Logic*, 74(3):1069–1080, 2009.

[FT08] S.-D. Friedman and K. Thompson. Perfect trees and elementary embeddings. *J. Symb. Logic*, 73(3):906–918, 2008.

[Git89] M. Gitik. The negation of singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$. *Ann. Pure Appl. Logic*, 43:209–234, 1989.

[GK09] M. Gitik and J. Krueger. Approachability at the second successor of a singular cardinal. *J. Symb. Logic*, 74(4):1211–1224, 2009.

[Ham09] J. D. Hamkins. Tall cardinals. *Math. Logic Quarterly*, 55(1):68–86, 2009.

[Hon19] R. Honzík. A Laver-like indestructibility for hypermeasurable cardinals. *Archive Math. Logic*, 58(3):257–287, 2019.

[HS85] L. Harrington and S. Shelah. Some exact equiconsistency results in set theory. *Notre Dame J. Formal Logic*, 26(2):178–188, 1985.

[HS20] R. Honzík and Š. Stejskalová. Indestructibility of the tree property. *J. Symb. Logic*, 85(1):467–485, 2020.

[HS22] R. Honzík and Š. Stejskalová. Small $\mathfrak{u}(\kappa)$ at singular $\kappa$ with compactness at $\kappa^{++}$. *Archive Math. Logic*, 61:33–54, 2022.

[HV16] R. Honzík and J. Verner. A lifting argument for the generalized Grigorieff forcing. *Notre Dame J. Formal Logic*, 57(2):221–231, 2016.

[Jech03] T. Jech. *Set Theory*. Springer Monographs in Mathematics. Springer, 2003.

[JS90] R. Jensen and K. Schlechta. Result on the generic Kurepa hypothesis. *Archive Math. Logic*, 30:13–27, 1990.

[Kan80] A. Kanamori. Perfect-set forcing for uncountable cardinals. *Ann. Math. Logic*, 19:97–114, 1980.

[Kan03] A. Kanamori. *The Higher Infinite*. Springer, 2003.

[Mag82] M. Magidor. Reflecting stationary sets. *J. Symb. Logic*, 47(4):755–771, 1982.

[Mit72] W. J. Mitchell. Aronszajn trees and the independence of the transfer property. *Ann. Math. Logic*, 5(1):21–46, 1972.

[RS19] A. Rosłanowski and S. Shelah. The last forcing standing with diamonds. *Fundamenta Mathematicae*, 246:109–159, 2019.

[She16] S. Shelah. *Proper and Improper Forcing*. Cambridge University Press, 2nd ed., 2016.

[Tod81] S. Todorčević. Some consequences of MA + ¬wKH. *Topology and its Applications*, 12:187–202, 1981.

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**PHILOSOPHICA ET HISTORICA 1** / MISCELLANEA LOGICA

# THE RABIN–KEISLER THEOREM AND THE SIZES OF ULTRAPOWERS

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**ABSTRACT**

Recall the Rabin–Keisler theorem which gives a lower bound $\kappa^\omega$ for the size of proper elementary extensions of complete structures of size $\kappa$, provided that $\kappa$ is an infinite cardinal below the first measurable cardinal. We survey—and at places clarify and extend—some facts which connect the Rabin–Keisler theorem, sizes of ultrapowers, combinatorial properties of ultrafilters, and large cardinals.

**Keywords:** Rabin–Keisler theorem; sizes of ultrapowers; non-regular ultrafilters.

## 1 Introduction

In this short survey, we gather some facts scattered in the literature which connect first-order theories, elementary extensions and ultrapowers. As a starting point we consider the following question:

> The Löwenheim–Skolem theorem (LS theorem for short) says that every infinite structure $M$ for a language $L$ has an elementary extension of every size greater or equal to $|L| + \aleph_0$. In particular, every theory $T$ with an infinite model has a model of every size greater or equal to $|L| + \aleph_0$. The question is whether the LS theorem really depends on $|L|$, or not.

On a quick look one might think that if $M$ is a countable structure in an uncountable language $L(M)$, then the language must be in some sense "trivial" (except for some countable sublanguage) if it can be realized on a countable domain. This idea might gain more plausibility by the loosely formulated fact that first-order theories are not strong enough to control infinite sizes, so if a theory $T$ has a model of size $\aleph_0$, it probably has models of every infinite size.

We will review below some folkore facts and results which show that this idea is false: the size of the language $|L|$ in the LS theorem is essential, and for example there is consistently a theory whose models exist in every infinite size except for $\aleph_1$ (in fact, this is a consequence of $2^{\aleph_0} = \aleph_2$ as we will see below). The bottom line is that first-order theories can control the sizes of their models provided these sizes are less or equal than the size of $|L| + \aleph_0$ (see the short paper [Mek77] for an example).

The paper is centered around the Rabin–Keisler theorem as stated for instance in [BS74, Theorem 5.6] or [Cha65]. We give this theorem as Theorem 3.11. This theorem marks the importance of the ultrapower construction in the model theory of the

https://www.doi.org/10.14712/24647055.2025.3

first-order predicate logic. It is interesting from several perspectives; we will focus on the fact that while it *a priori* does not deal with large cardinals, the very statement of the theorem for an arbitrary $\kappa$ needs the notion of a measurable cardinal (see Section 4.1). The connection to large cardinals is accentuated by more recent set-theoretic research which shows that the size of ultrapowers is closely connected to combinatorial properties of ultrafilters, which in turn often pre-suppose some large cardinals (see Section 4.3).

These results appeared in various books and papers, but are often written from different perspectives, without proper proofs and with different focus at different times (the results stretch over several decades). We briefly review some of these results using a unified notation with emphasis on the connections to modern set theory and large cardinals. We will also briefly comment on the question whether large cardinals are natural to logic (and mathematics) or they are artificial notions imported by set theory.

## 2 Proper elementary extensions, ultrafilters generated by "ideal" elements

Recall the standard method of defining an $\omega_1$-complete (normal) ultrafilter on a regular uncountable $\kappa$ using an elementary embedding (see the reference book [Kan03] for more details and also for the notational conventions): suppose $j : M \to N$ is an elementary embedding between transitive models of set theory $M, N$ (the language is just the language of set theory) such that the critical point of $j$ is a regular cardinal $\kappa$ and the powerset of $\kappa$ is a subset of $M$. Then it is easy to check that

$$U_j = \{ X \subseteq \kappa \,;\, \kappa \in j(X) \} \tag{1}$$

is an $\omega_1$-complete normal ultrafilter on $\kappa$. $U_j$ is generated by "ideal" element $\kappa$ (in the sense that $\kappa$ is not in the range of $j$). Since it is known that such ultrafilters imply consistency of ZFC (and much more), it follows that the existence of $j : M \to N$ as above cannot be proved in ZFC.

It may be surprising that this natural idea of defining an ultrafilter via an "ideal" element can be formulated also in the context of ZFC without any large cardinal strength: see the definition in (2) below. Without large cardinals, the construction will lose its easy formulation, but it is still useful.[1]

Suppose $T$ is a first-order theory in language $L(T)$ with an infinite model. With $L(T)$ given, we write $\lambda_T = |L(T)| + \aleph_0$. By the compactness theorem, it is easy to show that $T$ has models of every size $\geq \lambda_T$. Let us give some observations related to models of size $< \lambda_T$.

We first discuss these notions in the language of structures. The reformulation for theories is discussed in Remark 3.4.

If $A$ is an infinite structure, let $L(A)$ be the language of $A$.

[1]The reader will note that the LS theorem guarantees the existence of many embeddings like $j : M \to N$ (yielding the elementary extensions of $M$), but since the first-order logic is weak, it cannot guarantee that $N$ is *well-founded* (equivalently, *transitive*). This looks like a minor thing, but all of the large cardinal strength of $U_j$ comes from this fact.

**Definition 2.1** We say that $A$ is a *complete structure* if for every $a \in A$ there is a constant symbol $\dot{a}$ in $L(A)$ with $\dot{a}^A = a$, and for every $R \subseteq A^n$ where $1 \leq n < \omega$ there is a symbol $\dot{R}$ in $L(A)$ of the corresponding arity such that $(\dot{R})^A = R$.

It follows that the language of a complete structure has size $2^{|A|}$.

Suppose $A$ is a complete structure and $\mathrm{ProperExt}(A)$ is the set of proper elementary superstructures of $A$ in $L(A)$, i.e.

$$\mathrm{ProperExt}(A) = \{\, B\,;\; B \text{is in } L(A),\, A \subsetneq B \text{ and } A \prec B \,\}.$$

Let us consider the partial order $\preceq$ on $\mathrm{ProperExt}(A)$. As it turns out, the set of ultrapowers in $\mathrm{ProperExt}(A)$ is dense in the following sense (compare with [Kei71, Theorem 47]):

**Theorem 2.2** *Suppose* $B \in \mathrm{ProperExt}(A)$. *Then there is a non-principal ultrafilter* $U$ *such that* $\mathrm{Ult}(A^A, U) \in \mathrm{ProperExt}(A)$ *and* $\mathrm{Ult}(A^A, U) \preceq B$ *(up to isomorphism). If* $|A| = \omega$, *then* $U$ *is uniform.*

**Proof** Let $x$ be some fixed element in $B \setminus A$. Let us define

$$U = \{\, X \subseteq A\,;\; x \in \dot{X}^B \,\}, \text{where } X = \dot{X}^A. \tag{2}$$

$U$ contains for every $X \subseteq A$ either $X$ or $A \setminus X$: for $X \subseteq A$ and $Y = A \setminus X$, we have by elementarity that $\dot{X}^B \cup \dot{Y}^B = B$ and $\dot{X}^B \cap \dot{Y}^B = \emptyset$. The other properties of $U$ are verified similarly, and so $U$ is an ultrafilter. Let us consider $\mathrm{Ult}(A^A, U)$ and let us identity $[f_a]$ with $a$, where $f_a$ is a constant function with value $a$, so that $A \subseteq \mathrm{Ult}(A^A, U)$.

We cannot in general conclude that $U$ is uniform,[2] but $U$ is always non-principal in the sense that does not contain finite sets: for every finite subset $X = \{x_0, \dots, x_n\}$ of $A$, there is a first-order formula $\varphi_X$ which determines the elements of $X$.[3] It follows by elementarity applied to $\varphi_X$ that $\dot{X}^B = \dot{X}^A$, and hence $\dot{X}^B$ cannot contain the new element $x$. If $|A| = \omega$, then it immediately follows that $U$ is uniform.

The fact that $U$ is non-principal implies that $\mathrm{Ult}(A^A, U)$ is a proper elementary extension because the diagonal function $f(a) = a$ is different from every constant function (mod $U$).

We have verified $A \prec \mathrm{Ult}(A^A, U)$. Let us show $\mathrm{Ult}(A^A, U) \preceq B$. Let us define $h : \mathrm{Ult}(A^A, U) \to B$ by setting, for $\dot{f}^A = f$, $h([\dot{f}^A]) = \dot{f}^B(x)$. We have the following equalities:

$$\begin{aligned} \mathrm{Ult}(A^A, U) \models \varphi([f]) \Leftrightarrow \{\, a \in A\,;\; A \models \varphi(\dot{f}^A(a)) \,\} \in U \Leftrightarrow \\ x \in \dot{X}^B \Leftrightarrow B \models \varphi(\dot{f}^B(x)), \end{aligned} \tag{3}$$

where $\dot{X}$ is chosen to have $\dot{X}^A = \{a \in A; A \models \varphi(\dot{f}^A(a))\}$. We can take an isomorphic copy if necessary identify $\mathrm{Ult}(A^A, U)$ with an elementary submodel of $B$. □

[2] In fact, it may not be because $\mathrm{ProperExt}(A)$ can contain small structures which cannot be generated by uniform ultrafilters; see Section 3.4. However, if we are willing to go beyond the first-order logic, we can obtain uniform ultrafilters on uncountable cardinals; see Section 4.2 for more details.

[3] For instance $\varphi_X = (\forall x)(\dot{X}(x) \to x = \dot{x}_0 \vee \cdots \vee x = \dot{x}_n) \wedge (\dot{X}(\dot{x}_0) \wedge \cdots \wedge \dot{X}(\dot{x}_n))$.

## 3 Combinatorial properties of ultrafilters and sizes of elementary extensions

### 3.1 Uniform ultrafilters

It follows by Theorem 2.2 that the minimal size of a structure in $\mathrm{ProperExt}(A)$ for a complete $A$ is determined by the size of ultrapowers. For uniform ultrapowers there are some immediate lower bounds.

**Lemma 3.1** *Suppose $A$ is a complete structure of size $\kappa$ where $\kappa$ is an infinite cardinal.*

*(i) Assume $\kappa^{<\kappa} = \kappa$. Then for every uniform ultrafilter $U$ on $\kappa$, $\mathrm{Ult}(A^A, U)$ has size exactly $2^\kappa$.*

*(ii) Let there exist on $\kappa$ an almost-disjoint system $X$ of size $\lambda$ with $\kappa < \lambda \leq 2^\kappa$. Then for every uniform ultrafilter $U$ on $\kappa$, $\mathrm{Ult}(A^A, U)$ has size at least $\lambda$.*

**Proof** Let us first prove (ii). Let $X = \{ X_i \,;\, i < \lambda \}$ be an almost disjoint family of size $\lambda$ of subsets of $A$ (every $X_i$ has size $\kappa$ and $X_i \cap X_j$ has size $< \kappa$ for $i \neq j$). Let us fix for every $i$ some bijection $f_i : A \to X_i$. It follows that if $i \neq j$, then $\{ a \in A \,;\, f_i(a) = f_j(a) \}$ is bounded in $\kappa$. It follows $[f_i]_U \neq [f_j]_U$, and hence $h(i) = [f_i]_U$ is an injective function from $\lambda$ to $\mathrm{Ult}(A^A, U)$.

Claim (i) follows from claim (ii) by observing that $\kappa^{<\kappa} = \kappa$ implies that there exists an almost-disjoint system of size $2^\kappa$. □

Lemma 3.1 gives the following (because non-principal equals uniform for $\omega$):

**Corollary 3.2** (a version of the Rabin–Keisler theorem) *If $A$ is a complete structure of size $\omega$, then every element of $\mathrm{ProperExt}(A)$ has size at least $2^\omega$.*

Lemma 3.1 determines the size of ultrapowers via uniform ultrafilters in many situations, for instance under GCH:

**Corollary 3.3** *Suppose* GCH *holds, $\kappa$ is a regular cardinal, and $U$ is a uniform ultrafilter on $A$ with $|A| = \kappa$. Then $\mathrm{Ult}(A^A, U)$ has size $2^\kappa$.*

However, note that the ultrafilter $U$ from (2) may be non-uniform for uncountable $\kappa$, so Lemma 3.1 does not completely determine the least size of structures in $\mathrm{ProperExt}(A)$.

**Remark 3.4** Let $A$ be any complete structure. Let $T_A$ be the theory in the language $L(A)$ (including any language $A$ natively has) which contains all sentences which are true in $A$ in this extended language. Note that $A$ is a model $T_A$, and any other model is up to isomorphism in $\mathrm{ProperExt}(A)$. It follows by Corollary 3.2 that there exists a first-order theory $T$ with language of size $2^\omega$ which has a countable model, and every other model has size at least $2^\omega$. The theory $T$ may extend ZFC or any other theory as desired.

### 3.2 Regular ultrafilters

There is a combinatorial concept which is stronger than uniformity and which implies that the associated ultrapower has the maximal size without making the extra assumptions about almost disjoint families and their sizes (as in Lemma 3.1).

**Definition 3.5** Let $U$ be an ultrafilter on an infinite cardinal $\kappa$. We say that $U$ is *regular* if there is a family $\{ X_i \,;\, i < \kappa \}$ of pairwise distinct sets in $U$ such that every infinite subcollection of $\{ X_i \,;\, i < \kappa \}$ has an empty intersection.

Notice that one can say equivalently that $\{ X_i \,;\, i < \kappa \}$ is a witness for regularity if for every $j < \kappa$ the set

$$Z_j = \{ i < \kappa \,;\, j \in X_i \} \tag{4}$$

is finite. Also note that regularity immediately implies that $U$ is non-principal (does not contain a singleton).

Though it is not immediately clear, regularity implies uniformity:

**Lemma 3.6** *Suppose $U$ is a regular ultrafilter on an infinite $\kappa$. Then $U$ is uniform.*

**Proof** Suppose $U$ is regular and suppose for contradiction that $U$ contains some set of size $\mu < \kappa$; let us assume that $\mu \in U$. Let $\{ X_i \,;\, i < \kappa \}$ be some sets in $U$. We will show that this family doess not witness regularity. Suppose for contradiction it does. Consider the family $\{ X_i \cap \mu \,;\, i < \kappa \}$ which are also sets in $U$. If this set is of size $< \kappa$, it follows that there is some $X_i \cap \mu$ which is contained as a subset in $\kappa$-many $X_j$'s (and their intersection is therefore non-empty because it equals $X_i \cap \mu$), so $\{ X_i \,;\, i < \kappa \}$ does not witnesses regularity. If the set is of size $\kappa$, consider for every $\alpha < \mu$ the set $Z_\alpha$ of all $X_i \cap \mu$ which contain $\alpha$ as an element; by our assumption $\{ X_i \,;\, i < \kappa \}$ is a witness for regularity, and so this set must be finite; it follows that $\bigcup_\alpha Z_\alpha$ has size at most $\mu$, but this contradicts the fact that $\bigcup_\alpha Z_\alpha = \{ X_i \cap \mu \,;\, i < \kappa \}$ has size $\kappa$. $\square$

We now show that regular ultrafilters give large ultrapowers (we follow [Hod93, Theorem 9.5.4]).

**Theorem 3.7** *Let $U$ be a regular ultrafilter over some $A$ of size $\kappa$. Then $\mathrm{Ult}(A^A, U)$ has size $2^\kappa$.*

**Proof** We identify $A$ with $\kappa$ for easier reading. For every $j$, let $h_j : {}^{Z_j}A \to A$ be a bijection, where $Z_j$ is as in (4). For every $f : A \to A$, let $f^* : A \to A$ be defined as follows:

$$f^*(j) = h_j(f|Z_j).$$

A disagreement of $f$ and $g$ from $A$ to $A$ on a single argument translates into a disagreement of $f^*$ and $g^*$ on a set in $U$:

**Claim 3.8** *Suppose $f, g : A \to A$ and $f(i) \neq g(i)$ for some $i < \kappa$. Then $f^*$ and $g^*$ are different on all arguments $j$ in $X_i$ (where $X_i \in U$).*

**Proof** For every $Z_j$ such that $i \in Z_j$ it holds that $f|Z_j \neq g|Z_j$, and since $h_j$ is injective, we have $h_j(f|Z_j) \neq h_j(g|Z_j)$. Now notice that $j \in X_i$ implies $i \in Z_j$ for every $j$, and so the disagreement of $f^*$ and $g^*$ is witnessed on the whole set $X_i \in U$. $\square$

This shows that if $f \neq g$, then $[f^*]_U \neq [g^*]_U$ because $f^*$ and $g^*$ are different on a set in $U$. □

**Remark 3.9** In fact, the theorem gives information about the size of $\mathrm{Ult}(A^I, U)$ for infinite structure $A$ and regular ultrafilters on $I$: consider injections $h_j : {}^{Z_j}A \to A$ and functions $f$ from $I$ to $A$. Then the size of the ultrapower is $|A|^{|I|}$.

However, we do not get a generalization of the Rabin–Keisler theorem regarding the minimal size of structures in $\mathrm{ProperExt}(A)$ because even if all uniform ultrafilters are regular (which holds for instance in $V = L$, or more generally if we forbid some very large cardinals, see Section 4.3 for more details), there are always non-uniform ultrafilters which tend to have small ultrapowers, see Section 3.4.

### 3.3 Non-$\sigma$-complete ultrafilters

We saw that regular ultrafilters are always uniform and that this property makes them not general enough for the analysis of $\mathrm{ProperExt}(A)$. There a different concept, i.e. non-$\sigma$-completeness defined below which gives more information.

**Definition 3.10** An ultrafilter $U$ on an infinite cardinal $\kappa$ is called *$\sigma$-complete* if it is closed under the intersection of countably many sets in $U$. The same concept is also called *$\omega_1$-complete*. The extension of this concept to $\kappa$-completeness is obvious. $U$ is *non-$\sigma$-complete* if it is not $\sigma$-complete.

**Theorem 3.11** (Rabin–Keisler) *Let $\kappa$ be an infinite cardinal on which every non-principal ultrafilter is non-$\sigma$-complete. If $A$ is a complete structure of size $\kappa$, then every element of* $\mathrm{ProperExt}(A)$ *has size at least $\kappa^\omega$.*

Note that if $\kappa$ is inaccessible, then $\kappa^\omega = \Sigma_{\nu<\kappa}\nu^\omega = \kappa$, so the theorem does not say much regarding the size of elements in $\mathrm{ProperExt}(A)$ for an inaccessible $\kappa$. It has informational value if $\kappa$ satisfies the assumptions of the theorem and $\kappa$ is singular of countable cofinality (because in this case $\kappa^\omega > \kappa$), or with failures of GCH which increase the number of countable subsets of $\kappa$. For $\kappa = \omega$, it follows directly from an easier construction in Corollary 3.2.

First we show a version of the almost-disjointness property:

**Lemma 3.12** *Suppose $\kappa$ is an infinite cardinal. There is an almost disjoint family $X$ of size $\kappa^\omega$ of countable subsets of $\kappa$ (for $x \neq y \in X$, $|x \cap y| < \omega$).*

**Proof** This is a variant of the usual construction of an almost disjoint family of size $2^\kappa$ provided $2^{<\kappa} = \kappa$: it is enough to construct an almost disjoint family on $\kappa^{<\omega}$ and then use the bijection between $\kappa$ and $\kappa^{<\omega}$ to transfer it to $\kappa$. On $\kappa^{<\omega}$, the collection of cofinal branches $\kappa^\omega$ through $\kappa^{<\omega}$ viewed as a tree is an example of such a family. □

Let us now prove Theorem 3.11 (following [BS74, Theorem 5.4]):

**Proof** (Of Theorem 3.11) Suppose $A \prec B$ and $B$ is a proper extension. As in (2), define a non-principal ultrafilter $U$ determined by some fixed element $x \in B \setminus A$. Let

$\langle\, r_{f,n} \mid f \in {}^{\omega}\kappa, n < \omega \,\rangle$ be some enumeration of elements of $A$ with respect to some almost disjoint family $X$ from the previous Lemma 3.12: $r_{f,n}$ is the $n$-th element of the countable subset indexed by $f$.

Since $U$ is non-$\sigma$-complete, there is a strictly decreasing sequence $\langle\, F_n \mid n < \omega \,\rangle$ of sets in $U$ with the empty intersection (we may assume $F_0 = A$).

For every $f \in {}^{\omega}\kappa$, let us define a function $\tau_f : A \to A$ as follows

$$\tau_f(a) = r_{f,n}, \text{ iff } a \in F_n \setminus F_{n+1}. \tag{5}$$

Let us write $Y_n$ for $F_n \setminus F_{n+1}$. Clearly $\{\, a \in A\,;\, \tau_f(a) = \tau_g(a) \,\} \notin U$ for $f \neq g$: since $f$ and $g$ are almost disjoint, they can agree only on some finite number $n$ of arguments. By the definition of $\tau_f$ and $\tau_g$ it follows that $\{\, a \in A\,;\, \tau_f(a) = \tau_g(a) \,\}$ is contained in $Y_0 \cup \cdots \cup Y_n$, and this set is not in $U$.

This proves that for $f \neq g$, $[\tau_f]_U \neq [\tau_g]_U$, and hence $\mathrm{Ult}(A^A, U)$ and also $B$ have size at least $\kappa^{\omega}$. □

### 3.4 Non-uniform ultrafilters

Up to now, we discussed ultrafilters on $A$ which give large ultrapowers of $A$. We now observe that if we use non-uniform ultrafilters on $A$, or equivalently ultrapowers of $A$ with uniform ultrafilters on sets smaller than $A$, we (non-surprisingly) obtain smaller ultrapowers. Let us illustrate this case on the following example:

**Lemma 3.13** *Assume* CH. *Suppose $U$ on $\omega_1$ contains some countable set; without loss of generality assume $\omega \in U$. Then $\mathrm{Ult}(\omega_1^{\omega_1}, U)$ has size $\omega_1$.*

**Proof** Assume for contradiction that there is a family $W = \{\, f_\alpha\,;\, \alpha < \omega_2 \,\}$ of functions from $\omega_1$ into $\omega_1$ which are pairwise $U$-inequivalent. Since $U$ contains $\omega$, also $W|\omega = \{\, f_\alpha|\omega\,;\, \alpha < \omega_2 \,\}$ must be pairwise $U$-inequivalent,[4] so in particular pairwise distinct and so $W|\omega$ must have size $\omega_2$. But by CH, $|{}^{\omega}\omega_1| = \omega_1$, a contradiction. □

It follows that the Rabin–Keisler theorem does not directly generalize from $\omega$ to $\omega_1$ if we require just the non-principality of the ultrafilters:

**Corollary 3.14** *Assume* CH. *Suppose $A$ is a complete structure of size $\omega_1$. Then $\mathrm{ProperExt}(A)$ contains a proper elementary extension of $A$ of the form $\mathrm{Ult}(A^A, U)$ for some $U$ generated by a non-principal ultrafilter on $\omega_1$, and this has size $\omega_1$.*

**Proof** Let $U'$ be a non-principal ultrafilter on $\omega$. This is a centered system on $\omega_1$ and by Zorn's lemma extends into some non-principal ultrafilter on $\omega_1$. □

**Corollary 3.15** *More generally: if $A$ is a complete structure of size $\kappa$ and $\kappa^{\omega} = \kappa$, then there is a non-principal ultrafilter $U$ on $A$ generated by a countable set such that $\mathrm{Ult}(A^A, U)$ has size $\kappa$.*[5]

[4] For every $X \subseteq \omega_1$, $X \in U$ implies $X \cap \omega \in U$ because $\omega \in U$.

[5] Note that [BS74, Theorem 5.1] proves this by taking $\mathrm{Ult}(A^{\omega}, U)$ for a non-principal $U$ on $\omega$. Observing the connection with non-uniform ultrafilters allows one to work just with the ultrafilters on the domain of the structure.

This gives the statement of the full Rabin–Keisler theorem formulated as an equivalence, see for instance [BS74, Theorem 5.1].

# 4 Some connections with large cardinals

## 4.1 The limits of the Rabin–Keisler theorem

Theorem 3.11 can be stated with the notion of a measurable cardinal: if $\kappa$ is the least cardinal with a $\sigma$-complete non-principal ultrafilter, then $\kappa$ is in fact measurable, so the following is true:

**Theorem 4.1** (Rabin–Keisler, reformulation) *Suppose $\kappa$ is an infinite cardinal smaller than the least measurable cardinal. If $A$ is a complete structure of size $\kappa$, then every element of* $\mathrm{ProperExt}(A)$ *has size at least $\kappa^\omega$.*

As we discussed in the paragraph before the statement of Theorem 3.11, the theorem provides a non-trivial lower bound for singular cardinals $\kappa$ with countable cofinality or in cases with failures of GCH, provided that $\kappa$ is smaller than the first measurable.

It is a natural question whether the assumptions that $\kappa$ is smaller than then first measurable, or that there is no $\sigma$-complete ultrafilter on $\kappa$, are necessary. Surprisingly, not much is known about this problem; in particular the following seems open:

**Question 4.2** *Is it consistent that there is a singular cardinal $\kappa$ with countable cofinality such that for some complete structure $A$ of size $\kappa$, there is a proper elementary extension of $A$ of size $\kappa$?*

Note the following context for this question: if $\kappa$ is singular with countable cofinality, no uniform ultrafilter $U$ on $\kappa$ can be $\sigma$-complete. However, it can consistently happen (for instance if there is a strongly compact cardinal) that there is some $\lambda$, $\omega < \lambda < \kappa$, some non-principal non-uniform ultrafilter $U$ on $\kappa$ generated by a set of size $\lambda$, and $U$ is $\sigma$-complete. Little reflection shows that $\lambda$ must be greater or equal than the first measurable cardinal. Existence of such $U$ blocks the argument from the proof of Theorem 3.11 because it may be that the ultrafilter from Theorem 3.11 is $\sigma$-complete.

## 4.2 The Rabin–Keisler theorem and strongly compact cardinals

If we are willing to go beyond the first-order logic, then the Rabin–Keisler theorem generalizes to other cardinals.

Let us consider the logic $L_{\kappa,\kappa}$ which allows formulas of length $< \kappa$ with conjunctions and disjunctions of length $< \kappa$ and with quantifications of length $< \kappa$. We say that $\kappa$ is *strongly compact* if for every theory $T$ in $L_{\kappa,\kappa}$ (in an arbitrarily large signature), if every subtheory of $T$ with size $< \kappa$ has a model, then the whole theory $T$ has a model.

If $\kappa$ is strongly compact, then it is not difficult to check that the construction in Theorem 2.2 yields a non-principal $\kappa$-complete ultrafilter $U$. The $\kappa$-completeness plus non-principality implies that $U$ is uniform, and by Lemma 3.1 and the fact that strong compactness of $\kappa$ implies $\kappa^{<\kappa} = \kappa$, we know that $\mathrm{Ult}(A^A, U)$ has size $2^\kappa$. It follows we obtain the following theorem:

**Theorem 4.3** (Rabin–Keisler, for strongly compact cardinals) *Suppose $\kappa$ is strongly compact. If $A$ is a complete structure of size $\kappa$, then every element of* $\mathrm{ProperExt}(A)$ *(where elementary extensions are now considered in the infinitary logic $L_{\kappa,\kappa}$) has size at least $2^\kappa$.*

### 4.3 Non-regular ultrafilters

We saw in Theorem 3.7 that regular ultrafilters on $A$ give the maximal possible size of ultrapowers $\mathrm{Ult}(A^A, U)$. There is a natural question whether there are non-regular ultrafilters; in view of Lemma 3.6 every non-uniform ultrafilter is non-regular, so to avoid trivialities, we are interested in non-regular uniform ultrafilters.

Let us first give a two-parameter version of regularity:

**Definition 4.4** Suppose $\kappa$ is an infinite cardinal, and $\omega \leq \lambda < \mu \leq \kappa$ are cardinals. We say that $U$ is $(\mu, \lambda)$*-regular* if there $\mu$-many elements $\{ X_i \,;\, i < \mu \}$ from $U$ such that the intersection $\bigcap F$ of any subfamily $F \subseteq \{ X_i \,;\, i < \mu \}$ with $|F| = \lambda$ is empty.

It follows that if $U$ on $\kappa$ is regular according to Definition 3.5 then it is $(\kappa, \omega)$-regular. Note that every uniform $U$ on $\kappa$ is $(\kappa, \kappa)$-regular, so for nontrivial context, $\lambda$ must be smaller than $\kappa$. Lemma 3.5 generalizes as follows:

**Lemma 4.5** *Suppose $U$ is a $(\kappa, \lambda)$-regular ultrafilter on an infinite $\kappa$, with $\lambda < \kappa$. Then $U$ is uniform.*

**Proof** This is like the proof of Lemma 3.6 observing that in the second part of the proof, every $Z_\alpha$ has size $< \lambda$, and hence $\bigcup_\alpha Z_\alpha$ has size $< \kappa$, which gives a contradiction. □

The existence of uniform ultrafilters $U$ which are not $(\kappa, \lambda)$-regular for some $\lambda < \kappa$ has a very large consistency strength. On the other hand, it is true in $V = L$ (and other core models) that every uniform ultrafilter is regular. We will not review the relevant results here, but an interested reader can consult [Mag79, FMS88, Don88, SJ99] for more information (ordered chronologically).

For the purposes of this article, let us just comment on the relevance for the Rabin--Keisler theorem. As we mentioned, for uncountable structures, the construction from Theorem 2.2 can yield non-uniform ultrafilters, so there is no direct connection with uniform non-regular ultrafilters. However, we can still ask about the size of the ultrapower $\mathrm{Ult}(A^A, U)$. We saw in Lemma 3.1 that if $|A| = \kappa$, and $\kappa^{<\kappa} = \kappa$, then this ultrapower has alway the maximal size for a uniform $U$. Not much is known about other possibilities; for instance, the following seems open:

**Question 4.6** *Is it consistent that there is a uniform ultrafilter $U$ on $\omega_1$ such that for some complete structure $|A| = \omega_1$, $|\mathrm{Ult}(A^A, U)| < 2^{\omega_1}$?*

Note that for this to happen, $U$ must be non-regular, and it must hold $\omega_1 < 2^\omega < 2^{\omega_1}$ and every almost disjoint family on $\omega_1$ must have size $< 2^{\omega_1}$. By [DD03], a lower bound for the consistency strength of this configuration is an inaccessible stationary limit of measurable cardinals.

## 4.4 Some more general comments on large cardinals

This statement of Rabin–Keisler theorem, Theorems 3.11 and 4.1, raises a legitimate question regarding the status of measurable cardinals: if there are no measurable cardinals (for example if we assume $V = L$), then the Rabin–Keisler theorem holds for every $\kappa$. If measurable cardinals exist, the picture is less clear and not much is known as we already mentioned in the previous sections.

In the interest of simplicity—provided we think that mathematics should be such—it is tempting to assume there are no measurable cardinals. On second thought, this clean cut suffers from various technical drawbacks: forbidding measurable cardinals in $V$ does not by itself remove them from other transitive models of ZFC, so configurations like in Question 4.6 can still arise $V$ even if there are no large cardinals.[6] So for the "clean cut" we should in fact postulate that

$$\text{The theory } \mathsf{ZFC} + \mathbb{M} \text{ is inconsistent,} \tag{6}$$

where $\mathbb{M}$ denotes "there exists a measurable cardinal".

However, this is essentially a finitary statement whose postulation seems arbitrary and without a real mathematical reason. There is an extensive discussion (see for instance [FFMS00]) whether such a reason can be obtained in a weaker sense by considering certain set-theoretic axioms of wide consequence which decide the existence of a measurable cardinal either way. In the context of measurable cardinals, an axiom worth considering could be $V = L$ which implies $\neg\mathbb{M}$, i.e. (6) is weakened to a provable fact

$$\text{The theory } \mathsf{ZFC} + \mathsf{V} = \mathsf{L} + \mathbb{M} \text{ is inconsistent.} \tag{7}$$

Whether $V = L$ is good axiom cannot be decided without a larger context which we have not developed here, and there is no general consensus (see again [FFMS00] for more references and details).

It is equally interesting to ask whether we should postulate

$$\text{The theory } \mathsf{ZFC} + \mathbb{M} \text{ is consistent.} \tag{8}$$

There is the tendency to view this postulate as preferable over the negative (6): unlike (6), (8) can be refuted by a proof of contradiction from $\mathsf{ZFC} + \mathbb{M}$ if there is one, whereas by Gödel's theorem there is no chance to refute (6).[7]

These few comments might suggest that we should not artificially "remove" the problem of measurability from the Rabin–Keisler theorem because there are no real mathematical reasons for doing so.

**Acknowledgement** The author was supported by GAČR grant *The role of set theory in modern mathematics* (24-12141S).

[6] The typical "trick" is to start with a universe $V$ where some property like in Question 4.6 holds: If $V$ has no large cardinals, we are done. If it has large cardinals, cut $V$ at the first inaccessible cardinal $\kappa$. Then $V_\kappa$ is a model of ZFC with no large cardinals, and yet the property holds because it concerns only sets low in the cumulative hierarchy.

[7] This is a fine distinction because refuting (8) is the same as verifying (6); but for general methodological reasons it is usually preferable to consider axioms which are in principle refutable over those which can be only verified, but never refuted, if verification is considered unlikely.

## References


[BS74] J. L. Bell and A. B. Slomson. *Models and ultraproducts: an introduction*. North-Holland / American Elsevier, 3rd ed., 1974.

[Cha65] C. C. Chang. A simple proof of the Rabin-Keisler theorem. *Bull. of the Amer. Math. Soc.*, 71:642–643, 1965.

[DD03] O. Deiser and H.-D. Donder. Canonical functions, non-regular ultrafilters and Ulam's problem on $\omega_1$. *J. Symb. Logic*, 68(3):713–739, 2003.

[Don88] H.-D. Donder. Regularity of ultrafilters and the core model. *Israel J. Math.*, 63(3):289–322, 1988.

[FFMS00] S. Feferman, H. M. Friedman, P. Maddy, and J. R. Steel. Does mathematics need new axioms? *Bull. Symb. Logic*, 6(4):401–446, 2000.

[FMS88] M. Foreman, M. Magidor, and S. Shelah. Martin's maximum, saturated ideals and non-regular ultrafilters. part II. *Ann. Mathematics*, 127(3):521–545, 1988.

[Hod93] W. Hodges. *Model theory*. Cambridge University Press, 1993.

[Kan03] A. Kanamori. *The Higher Infinite*. Springer, 2003.

[Kei71] H. J. Keisler. *Model theory for infinitary logic*. North-Holland, 1971.

[Mag79] M. Magidor. On the existence of nonregular ultrafilters and the cardinality of ultrapowers. *Trans. Amer. Math. Soc.*, 249(1):97–111, 1979.

[Mek77] A. Mekler. Theories with models of prescribed cardinalities. *J. Symb. Logic*, 42(2):251–253, 1977.

[SJ99] S. Shelah and R. Jin. Possible size of an ultrapower of $\omega$. *Archive Math. Logic*, 38:61–77, 1999.

2022 ACTA UNIVERSITATIS CAROLINAE PP. 57–75
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# SOME NOTES ON EMBEDDINGS, PROJECTIONS, AND EASTON'S LEMMA

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**ABSTRACT**

We survey some lesser-known facts concerning properties of embeddings and projections between forcing notions. We will also state some generalizations of Easton's lemma. To our knowledge, many of these facts have not been published, so we include their proofs for the benefit of the reader.

**Keywords:** forcing; forcing notion; dense embedding; regular embedding; complete embedding; projection; chain condition; closure.

## 1 Introduction

The method of forcing was introduced by Paul Cohen [Coh63, Coh64] in his proof of the independence of the axiom of choice and the continuum hypothesis over ZFC. Since then forcing has proved to be a powerful technique for producing consistency results.

A forcing notion is a partially ordered set $(P, \leq)$ with a greatest element. A substantial part of the forcing machinery deals with combinatorial properties of partially ordered sets. We will survey some results in this area; they are mostly combinatorial and require little knowledge of the forcing method but we do give some more details and definitions in Section 2.

In forcing constructions we often need to compare two forcing notions to find out whether they give rise to the same generic extension or whether one forcing notion gives rise to an extension which is smaller than the other one:

Suppose $P$ and $Q$ are two forcing notions. Does it hold that

> $(*)$ for each $P$-generic $G$ over $V$ there exists a $Q$-generic $H$ over $V$ in $V[G]$ such that $V[G] = V[H]$, and conversely?

This question is related to the notion of *forcing equivalence*, which is usually formulated more restrictively than $(*)$, see Definition 3.1. The definition of forcing equivalence is tightly connected to the notion of *dense embedding*. There are several non-equivalent and equivalent definitions of forcing equivalence and some strengthenings which use the notion of dense embedding. We survey some lesser-known facts related to these notions.

A natural weakening of $(*)$ is to ask whether for every $P$-generic filter $G$ over $V$, there is a $Q$-generic filter $H$ over $V$ in $V[G]$, yielding $V[H] \subseteq V[G]$. This question leads to the notions of *complete embedding* and *projection* between forcing notions,

https://www.doi.org/10.14712/24647055.2025.4

functions from $P$ to $Q$ or conversely with some extra properties. Existence of such functions makes it possible to view $P$ as a two-step iteration which starts with $Q$ and is followed by some other forcing notion which we call the quotient forcing (determined by $P$ and $Q$). In terms of forcing equivalence, $P$ is forcing equivalent to $Q * \dot{R}$, where $\dot{R}$ is a $Q$-name for the quotient forcing.

In the last section we discuss the chain condition, closure, and distributivity of forcing notions and their preservation by some other forcing notions. We will state some useful variations on Easton's lemma which feature more than two forcing notions and deal with distributivity.

## 2 Preliminaries

In this section we review some basic facts about forcing and fix notational conventions. The general reference is Jech's book [Jech03]; the treatment of the iteration of forcing notions follows Baumgartner's paper [Bau83].

A forcing notion is a partially ordered set $(P, \leq)$ with a greatest element, which we denote $1_P$. To simplify notation, we will often write $P$ instead of $(P, \leq)$ if the ordering is clear from the context.

A condition $p$ is stronger then $q$, in symbols $p \leq q$, if it carries more information. We say that two condition $p$ and $q$ are compatible, in symbols $p \,||\, q$, if there is an element of the ordering such that it is below both $p$ and $q$. We say that they are incompatible, if they are not compatible and we denote this by $p \perp q$. We say that $A \subseteq P$ is an *antichain* if all distinct $p, q$ in $A$ are incompatible; an antichain is *maximal* if every $p$ in $P$ is compatible with some element in $A$.

If $(P, \leq)$ is a forcing notion, we write $V[P]$ to denote a generic extension by $P$ if the concrete generic filter is not important. Sometimes we write $P \Vdash \varphi$ in place of $1_P \Vdash \varphi$.

We say that $(P, \leq)$ is *separative* if $p \not\leq q$ implies that there is some $r \leq p$ which is incompatible with $q$. Note that if $(P, \leq)$ is separative, then $p \leq q$ is equivalent to $p$ forcing $q$ into the generic filter.

A forcing notion is said to be *non-trivial* if below every condition there are two incompatible extensions. Otherwise the forcing notion is called *trivial*. Note that if $(P, \leq)$ is non-trivial, then any $P$-generic filter cannot be an element of the universe.

To obtain all generic extensions it suffices to consider only the separative orders: If $(P, \leq)$ is not separative, then it has a separative quotient which produces the same generic extensions as $P$. For more details about separative quotients see [Jech03].

Now we define the notion of *a lottery sum* of forcing notions to provide some counterexamples in Section 3. The concept of a "sum" of forcing notions has been around for a long time; for more details see [Ham00].

**Definition 2.1** Let $\{ P_i \,;\, i \in I \}$ be an indexed set of forcing notions $(P_i, \leq_{P_i})$. We define *the lottery sum*

$$\bigoplus \{ P_i \,;\, i \in I \} \tag{1}$$

as a forcing notion as follows: The underlying set is $\{ (i,p) \,;\, p \in P_i \;\&\; i \in I \} \cup \{1\}$ where $1$ is not an element of $\bigcup \{ P_i \,;\, i \in I \}$, the ordering is such that $1$ is the greatest element, and $(i,p) \leq (j,q) \leftrightarrow i = j$ and $p \leq_{P_i} q$.

The intuition is that a $\bigoplus\{ P_i \,;\, i \in I \}$-generic first chooses a forcing notion from $\{ P_i \,;\, i \in I \}$ to force with, and then forces with it.

Finally, we define several forcing notions which we will use to illustrate certain concepts in the following sections.

Cohen forcing is the forcing used by Cohen to show the independence of the continuum hypothesis [Coh63, Coh64].

**Definition 2.2** Let $\kappa \geq \omega$ be a regular cardinal and $\alpha > 0$ an ordinal. *Cohen forcing* at $\kappa$ of length $\alpha$, denoted by $\mathrm{Add}(\kappa, \alpha)$, is the set of all partial functions from $\kappa \times \alpha$ to 2 of size less than $\kappa$. The ordering is by reverse inclusion, i.e. $p \leq q \leftrightarrow q \subseteq p$.

Cohen forcing at $\kappa$ is $\kappa$-closed, and if $\kappa^{<\kappa} = \kappa$, then it is also $\kappa^+$-Knaster (see Definition 4.1).

The following forcing was introduced for $\kappa = \omega$ by Sacks in [Sac71] and the generalized version for a regular cardinal $\kappa > \omega$ was introduced by Kanamori [Kan80].

**Definition 2.3** Let $\kappa \geq \omega$ be a regular cardinal. We say that a set $(T, \subseteq)$ is a $\kappa$-perfect tree if the following hold:

(i) $T \subseteq {}^{<\kappa}2$ and $T$ is closed under initial segments, i.e. if $t \in T$ and $s \in {}^{<\kappa}2$ is such that $s \subseteq t$, then $s \in T$;

(ii) $\forall t{\in}T \exists s{\in}T (t \subseteq s \;\&\; s^\frown 0 \in T \;\&\; s^\frown 1 \in T)$, that is, above every node $t \in T$ there is a splitting node $s$;

(iii) If $\langle\, s_\alpha \mid \alpha < \gamma \,\rangle$ for $\gamma < \kappa$ is a $\subseteq$-increasing sequence of nodes in $T$, then the union $s = \bigcup_{\alpha<\gamma} s_\alpha$ is in $T$;

(iv) If there are unboundedly many splitting nodes below $s \in T$, then $s$ splits, i.e. if for every $t \subset s$ there exists a splitting node $t'$ such that $t \subset t' \subset s$, then $s$ splits in $T$.

Note that if $\kappa = \omega$ the items (iii) and (iv) are redundant.

**Definition 2.4** Let $\kappa \geq \omega$ be a regular cardinal. *Sacks forcing* at $\kappa$, $\mathrm{Sacks}(\kappa, 1)$, is the collection of all $\kappa$-perfect trees as in the previous definition. The ordering is by inclusion, i.e. $p \leq q \leftrightarrow p \subseteq q$.

**Remark 2.5** For $\kappa > \omega$, we can change the item (iv) in Definition 2.3 in various ways. For example we can require that the item (iv) holds only for nodes of a given fixed cofinality and forbid the splitting on other cofinalities, see [FH12]. Or in general we can require item (iv) only for some stationary subset $S$ of $\kappa$; i.e. if there are unboundedly many splitting nodes below $s \in T$ and the height of $s$ is in $S$, then $s$ splits. We can also add some additional properties regarding the splitting nodes with respect to some stationary subset of $\kappa$, see Definition 3.1 (3) in [JS01]. These modifications provide variations of the Sacks forcing with some additional properties.

Now, we define a forcing for adding a closed unbounded subset to a stationary subset of $\omega_1$, which is due to Baumgartner, Harrington and Kleinberg [BHK76].

**Definition 2.6** Let $S \subseteq \omega_1$ be stationary. We define a forcing $\mathrm{CU}(S)$ which adds a closed unbounded set to $S$. The conditions in $\mathrm{CU}(S)$ are closed bounded subset of $S$ ordered by end-extension.

Note that we can define the forcing notion $\mathrm{CU}(X)$ for every subset $X$ of $\omega_1$. However, if $X$ is not stationary, then the forcing $\mathrm{CU}(X)$ collapses cardinals. More precisely, $\mathrm{CU}(S)$ is $\omega_1$-distributive (see Definition 4.1) if and only if $S$ is stationary. If $S \subseteq \omega_1$ is stationary and co-stationary (i.e. $\omega_1 \setminus S$ is stationary) then forcing with $\mathrm{CU}(\omega_1 \setminus S)$ destroys the stationarity of $S$.

**Definition 2.7** Let $\kappa > \omega$ be a regular cardinal. We say that a $\kappa$-tree is a *$\kappa$-Suslin tree* if it has no cofinal branches and does not contain antichains of size $\kappa$.

When forcing with a tree $T$, the ordering is the reverse ordering of the tree $T$. A $\kappa$-Suslin tree viewed as a forcing notion is $\kappa$-cc and $\kappa$-distributive (see Definition 4.1), in particular forcing with a Suslin tree preserves all cardinals.

In contrast to the forcing notions mentioned so far, $\kappa$-Suslin trees exist only consistently. For example, under $\mathsf{MA}_{\aleph_1}$ (Martin's Axiom) there are no $\omega_1$-Suslin trees; on the other hand, under the assumption of $\diamondsuit$, there are always $\omega_1$-Suslin trees. Sometimes it is convenient to consider Suslin trees with some additional properties:

**Definition 2.8** Assume that $T$ is a tree and $s$ is in $T$. Let $T_s$ denote the set of all nodes in $T$ which are comparable with $s$; i.e. $T_s = \{\, t \in T \,;\, t \leq_T s \ \vee\ s \leq_T t \,\}$.

**Definition 2.9** Let $S$ and $T$ be trees of height $\omega_1$. Let $S \otimes T$ denote the set of all pairs $(s,t)$ such that there is an ordinal $\gamma < \omega_1$ with $s \in S_\gamma$ and $t \in T_\gamma$. The ordering of $S \otimes T$ is component-wise: $(s,t) <_{S\otimes T} (s',t')$ if $s <_S s'$ and $t <_T t'$.

**Definition 2.10** Let $T$ be an $\omega_1$-tree and let $0 < n < \omega$. A *derived tree of dimension* $n$ (or an *$n$-derived tree*) is a tree of the form

$$T_{t_0} \otimes T_{t_1} \otimes \cdots \otimes T_{t_{n-1}}, \tag{2}$$

where $t_0, \ldots, t_{n-1}$ are distinct elements of $T$ of the same height.

A derived tree of dimension 1 is just a tree of the form $T_t$ where $t \in T$ .

**Definition 2.11** Let $1 \leq n < \omega$. A Suslin tree $T$ is *$n$-free* if all of its $n$-derived trees are Suslin. A Suslin tree $T$ is *free* if it is $n$-free for all $1 \leq n < \omega$.

Free Suslin trees were originally introduced in [Jen] by Jensen under the name full Suslin trees.

**Definition 2.12** An $\omega_1$-tree $T$ is *rigid* if there does not exist any automorphism of $T$ other than the identity function. It is *homogeneous* if for all $t$ and $s$ in $T$ with the same height, there exists an automorphism $f : T \to T$ such that $f(t) = s$.

Free $\omega_1$-Suslin trees are rigid. Free and homogeneous $\omega_1$-Suslin trees can be constructed from $\diamondsuit$ (the construction is due to Jensen).

# 3 Comparing forcing notions

In this section we state some facts concerning the comparison of forcing notions. To our knowledge, many of these facts have not been written up in detail in literature, so we include their proofs for the benefit of the reader. The books [Kun80] and [Abr10] are a general reference for this section.

For the purposes of this section, we assume (unless we say otherwise) that our forcing notions are non-trivial and separative.

## 3.1 Forcing equivalence and dense embeddings

Recall that if $(Q, \leq_Q)$ is a partial order, then we can find a complete Boolean algebra $(\mathrm{RO}(Q), \leq_{\mathrm{RO}(Q)})$ and a dense embedding $i$ from $Q$ to the positive part $\mathrm{RO}^+(Q)$ of $\mathrm{RO}(Q)$, i.e. to the set $\{\, b \in \mathrm{RO}(Q) \;;\; b > 0_{\mathrm{RO}(Q)} \,\}$. The algebra $\mathrm{RO}(Q)$ is unique up to isomorphism. If $(Q, \leq_Q)$ is in addition separative, then the mapping $i$ is 1-1 and hence it is an isomorphism between $Q$ and some dense subset of $\mathrm{RO}^+(Q)$; in this case we identify $Q$ with a dense subset of $\mathrm{RO}^+(Q)$ when we work with the Boolean completion of $Q$.

The uniqueness of the Boolean completion can be used to define a natural notion of *forcing equivalence* of forcing notions:

**Definition 3.1** We say that two forcing notions $(P, \leq_P)$ and $(Q, \leq_Q)$ are *forcing equivalent* if their Boolean completions are isomorphic.

It is easy to see that forcing-equivalence implies the following weaker model-theoretic property:

> $(*)$ for any $P$-generic $G$ over $V$ there exists a $Q$-generic $H$ over $V$ in $V[G]$ such that $V[G] = V[H]$, and conversely.

If $P$ is any forcing notion, then the lottery sum of $\kappa$-many copies of $P$ for $\kappa \geq (2^{|P|})^+$ yields a non-equivalent forcing notion which however satisfies the model-theoretic condition $(*)$.

We will discuss several concepts related to the relationship between two forcing notions $(P, \leq_P)$ and $(Q, \leq_Q)$; these concepts will be formulated in terms of the existence of certain functions from $P$ to $Q$ (and conversely) and also in terms of model-theoretic conditions which are weakenings of the condition $(*)$.

**Definition 3.2** We say that a function $i : P \to Q$ between partial orders $(P, \leq_P)$ and $(Q, \leq_Q)$ is a *dense embedding* if it is order-preserving, $i(p) \perp i(p')$ whenever $p \perp p'$, and the range of $i$ is dense in $Q$.

It is easy to check that the existence of a dense embedding implies forcing equivalence, but the converse does not necessarily hold. In fact, we will show below that forcing equivalence does not even imply a weaker condition than the existence of a dense embedding; this weaker condition is stated in Lemma 3.4.

Let us state two lemmas (Lemma 3.3 and 3.4) which are used in practice to check that two forcing notions are equivalent. In both cases, there is a third forcing notion which is used to compare the two. The first lemma provides an equivalent characterization while the second one gives only a sufficient condition. The proofs are an exercise for the reader.

**Lemma 3.3** *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be forcing notions. The following are equivalent:*

*(i)* *$(P, \leq_P)$ and $(Q, \leq_Q)$ are forcing equivalent;*

*(ii)* *There exists a forcing notion $(S, \leq_S)$ such that both $(P, \leq_P)$ and $(Q, \leq_Q)$ densely embed into $(S, \leq_S)$.*

Instead of $P, Q$ densely embedding into $S$, we may consider the opposite configuration with $S$ densely embedding into $P, Q$:

**Lemma 3.4** *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be forcing notions. If there exists a forcing notion $(S, \leq_S)$ such that $(S, \leq_S)$ densely embeds into both $(P, \leq_P)$ and $(Q, \leq_Q)$, then the notions $(P, \leq_P)$ and $(Q, \leq_Q)$ are forcing equivalent.*

Notice that Lemma 3.4 gives only a sufficient condition for forcing equivalence. In fact, we will show that the converse of Lemma 3.4 does not hold in general. To find a counterexample, it suffices to consider a forcing notion $R$ with the property that if we force with $R$, there will be only one generic filter over $R$ in the generic extension by $R$. More precisely, if $G$ is $R$-generic over $V$ and $H$ is $R$-generic in $V[G]$, then $G = H$. If this holds, we say that $R$ has the *unique generic property*.

If $R$ has the unique generic property, then any two disjoint dense subsets $P$ and $Q$ of $R$ will give a counterexample to the converse of Lemma 3.4:

**Lemma 3.5** *Let $(R, \leq)$ be a forcing notion*[1] *and let $P, Q \subseteq R$ be two disjoint dense subsets of $R$. Moreover, assume that there is a forcing notion $(S, \leq_S)$ with dense embeddings $i : S \to P$ and $j : S \to Q$. Then for every $s \in S$ there is a $t \in S$ with $t \leq s$ such that $i(t) \perp j(t)$ in $R$.*

**Proof** Let $s \in S$ be arbitrary. Since $P$ and $Q$ are disjoint, we must have $i(s) \neq j(s)$, and in particular $j(s) \not\leq i(s)$ or $i(s) \not\leq j(s)$. Assume without loss of generality that $j(s) \not\leq i(s)$; by separativity, there is $r \in R$ such that $r \leq j(s)$ and $r \perp i(s)$. Since $P$ is dense in $R$, there is $p \in P$ such that $p \leq r \leq j(s)$. Note that $p \perp i(s)$. Since $p \in P \subseteq R$ and $Q$ is dense in $R$ there is $q \in Q$ such that $q \leq p \leq j(s)$. Since $j$ is a dense embedding, there is $t \leq s$ in $S$ such that $j(t) \leq q \leq p \leq j(s)$. But now $i(t) \leq i(s) \perp p$, and hence $i(t) \perp p$ and $i(t) \perp j(t)$. □

It follows that if $P, Q, S, R$ are as in Lemma 3.5, then $R$ cannot have the unique generic property: If $G$ is $S$-generic, then $H_0 = i[G]$ and $H_1 = j[G]$ generate two generic filters over $R$ which must be different (the set of the $t$'s with $j(t) \perp i(t)$ is dense in $S$).

This leaves us with the question whether there is a forcing $R$ with the unique generic property. One well-known example is a 2-free Suslin tree; see Definition 2.11 above for more details. There is also a more complicated example in ZFC, constructed by Jech and Shelah in [JS01] using a variant of the Sacks forcing at an uncountable regular $\kappa$.

[1] Recall that we assume that forcing notions are separative.

### 3.2 Projections, complete embeddings and regular embeddings

Let us now turn to analyzing forcing notions $P, Q$ with $P$ giving a "bigger" extension than $Q$.

**Definition 3.6** We say that a function $\pi : P \to Q$ between $(P, \leq_P)$ and $(Q, \leq_Q)$ is a *projection* if it is order-preserving, $\pi(1_P) = 1_Q$, and

$$\text{for all } p \in P \text{ and all } q \leq_Q \pi(p) \text{ there is } p' \leq_P p \text{ such that } \pi(p') \leq_Q q.^2 \tag{3}$$

Let $\pi$ be as above and fix a $P$-generic filter $G$. If $D \subseteq Q$ is open dense in $Q$ then $\pi^{-1}{}''D$ is open dense in $P$ and it is easy to see that $\pi''G$ generates a $Q$-generic filter. Let us denote this generic filter by $H$.

The forcing $P$ can be decomposed into a two-step iteration of $Q$ followed by a quotient forcing $P/H$ defined as follows:

$$P/H = \{\, p \in P \,;\, \pi(p) \in H \,\}. \tag{4}$$

Now, it holds that $G$ is a $P/H$-generic filter over $V[H]$ and $V[G] = V[H][G]$, where in the first model $G$ is taken as a $P$-generic filter over $V$ and in the second as a $P/H$-generic filter over $V[H]$.

The converse holds as well. If we first take a $Q$-generic filter $H$ over $V$ and then a $P/H$-generic filter $G$ over $V[H]$, then $G$ is a $P$-generic filter over $V$ and moreover the generic filter $H$ is generated by $\pi''G$.

**Definition 3.7** We say that a function $i : Q \to P$ between partial orders $(Q, \leq_Q)$ and $(P, \leq_P)$ is a *complete embedding* if it is order-preserving, $i(q) \perp i(q')$ whenever $q \perp q'$ and

$$\text{for all } p \in P \text{ there is } q \in Q \text{ such that for all } q' \leq q,\ i(q') \,||\, p. \tag{5}$$

Analogues of facts mentioned for projections following Definition 3.6 hold also for complete embeddings. Let $i$ be as in the definition above and fix a $P$-generic filter $G$. If $D \subseteq Q$ is predense in $Q$ then $i''D$ is predense in $P$ and $i^{-1}{}''G$ is a $Q$-generic filter. Let us denote this generic filter by $H$ and in $V[H]$ define a quotient forcing as follows:

$$P/H = \{\, p \in P \,;\, \forall q \in H (p \,||\, i(q)) \,\}. \tag{6}$$

Then $G$ is a $P/H$-generic filter over $V[H]$ and $V[G] = V[H][G]$, where in the first model $G$ is taken as a $P$-generic over $V$ and in the second as a $P/H$-generic over $V[H]$.

The converse direction holds as well. If we first take a $Q$-generic filter $H$ over $V$ and define the quotient forcing $P/H$ and then take a $P/H$-generic filter $G$ over $V[H]$, then $G$ is $P$-generic over $V$ and moreover the generic filter $H$ is equal to $i^{-1}{}''G$.

**Remark 3.8** In general, the quotient forcings (4) and (6) of two separative forcings do not have to be separative. Consider the following easy example using Cohen forcing $\mathrm{Add}(\kappa, \alpha)$ (see Definition 2.2). Let $\kappa$ be a regular cardinal and $0 < \beta < \alpha$ be ordinals.

[2] Note that the condition $\pi(1_P) = 1_Q$ together with (3) ensure that the range of $\pi$ is dense in $Q$.

Then it is easy to see that $\pi : \mathrm{Add}(\kappa, \alpha) \to \mathrm{Add}(\kappa, \beta)$ defined by $\pi(p) = p \restriction (\kappa \times \beta)$ is a projection. Let $G$ be an $\mathrm{Add}(\kappa, \beta)$-generic filter over $V$. Then

$$\mathrm{Add}(\kappa, \alpha)/G = \{ p \in \mathrm{Add}(\kappa, \alpha) \, ; \, p \restriction (\kappa \times \beta) \in G \}. \tag{7}$$

It follows that all conditions in $\mathrm{Add}(\kappa, \beta)$ which are in $G$ are in $\mathrm{Add}(\kappa, \alpha)/G$ and also every condition $p$ in the quotient $\mathrm{Add}(\kappa, \alpha)/G$ is compatible with all conditions in $G$. Thus two arbitrary conditions $q_0 \neq q_1$ in $G$ witness that $\mathrm{Add}(\kappa, \alpha)/G$ is not separative. This argument can be modified for complete embeddings as well.

Complete embeddings have the following equivalent—and often more useful—characterization.

**Definition 3.9** We say that a function $i : Q \to P$ between partial orders $(Q, \leq_Q)$ and $(P, \leq_P)$ is a *regular embedding* if it is order-preserving, $i(q) \perp i(q')$ whenever $q \perp q'$, and $i''A$ is a maximal antichain in $P$, whenever $A$ is a maximal antichain in $Q$.

**Lemma 3.10** *Let $(Q, \leq_Q)$ and $(P, \leq_P)$ be two partial orders. Then a function $i$ from $Q$ to $P$ is a complete embedding if and only if it is a regular embedding.*

**Proof** Assume that $i$ is a complete embedding from $Q$ into $P$. Let $A \subseteq Q$ be a maximal antichain and let $p$ in $P$ be given. We will show that there is $a \in A$ such that $i(a) \parallel p$, hence $i''A$ is maximal. As $p$ is in $P$ there is $q \in Q$ such that for all $q' \leq q$, $i(q') \parallel p$ by (5). Since $A$ is maximal in $Q$, there is $a \in A$ such that $a \parallel q$, hence there is $q' \leq q$ such that $q' \leq a$. Therefore $i(q') \leq i(a)$ and $i(q') \parallel p$. Hence $i(a) \parallel p$.

For the converse direction assume that $i$ is a regular embedding between $Q$ and $P$. Let $p$ in $P$ be given and assume for contradiction that for all $q \in Q$ there is a $q' \leq q$ such that $i(q') \perp p$. Then the set

$$D = \{ q \in Q \, ; \, i(q) \perp p \} \tag{8}$$

is dense in $Q$. Let $A \subseteq D$ be a maximal antichain. Then, by the definition of a regular embedding $i''A$ is maximal in $P$, hence there exists $a \in A$ such that $i(a) \parallel p$. This is a contradiction as $a$ is also in $D$ and therefore $i(a) \perp p$. □

It would be tempting to claim that a projection from $(P, \leq_P)$ to $(Q, \leq_Q)$ ensures the existence of a complete embedding from $(Q, \leq_Q)$ to $(P, \leq_P)$ and conversely. But in general we need to use the Boolean completions of $P$ and $Q$.

**Lemma 3.11** *Let $(Q, \leq_Q)$ and $(P, \leq_P)$ be two partial orders. Then the following hold:*

*(i) If there is a complete embedding from $Q$ to $P$, then there is a projection from $P$ to $\mathrm{RO}^+(Q)$.*

*(ii) If there is a projection from $P$ to $Q$, then there is a complete embedding from $Q$ to $\mathrm{RO}^+(P)$.*

**Proof** (i). Let $i$ be a complete embedding from $Q$ to $P$. Let us define a function $\pi$ from $P$ to $\mathrm{RO}^+(Q)$ by

$$\pi(p) = \bigvee\{\, q \in Q \;;\; \forall q' \leq q (i(q') \mid\mid p) \,\}. \tag{9}$$

First note that $\pi$ is well-defined correctly for all $p \in P$ by (5). Moreover, for all $q$ in $Q$ it holds that

$$\pi(i(q)) = q. \tag{10}$$

To verify (10) denote $Q_p = \{\, q \in Q \;;\; \forall q' \leq q \; (i(q') \mid\mid p) \,\}$ for $p \in P$. Let us first show that $\pi(i(q)) \leq q$, i.e that for all $q^* \in Q_{i(q)}$ it holds that $q^*$ is below $q$: if not, there exists $q' \leq q^*$, which is incompatible with $q$, by separativity of $Q$; however, as $i$ is a complete embedding, it holds that $i(q') \perp i(q)$, which contradicts $q^*$ being in $Q_{i(q)}$. To show $\pi(i(q)) \geq q$, notice that for every $q' \leq q$ it holds that $i(q') \leq i(q)$; therefore $q$ is in $Q_{i(q)}$.

Let us now argue that $\pi$ is a projection. The order-preservation follows since $Q_{p'} \subseteq Q_p$ whenever $p' \leq p$. Since all conditions are compatible with the condition $1_{\mathrm{RO}^+(Q)}$, we have $\pi(1_P) = 1_{\mathrm{RO}^+(Q)}$.

Let us now prove condition (3). Assume that $b < \pi(p)$ (if $b = \pi(p)$ the condition is satisfied trivially). Since $\pi(p) = \bigvee\{\, q \in Q \;;\; \forall q' \leq q \; (i(q') \mid\mid p) \,\}$, there is $q \in Q$ such that $q \leq b$ and $i(q)$ is compatible with $p$. Hence there is $p^* \in P$ below both $i(q)$ and $p$. The rest now follows as $\pi(p^*) \leq \pi(i(q))$ and $\pi(i(q)) = q$ by (10).

(ii). Let $\pi$ be a projection from $P$ to $Q$. Let us define a function $i$ from $Q$ to $\mathrm{RO}^+(P)$ by

$$i(q) = \bigvee\{\, p \in P \;;\; \pi(p) \leq q \,\}. \tag{11}$$

First note that $i$ is well-defined for all $q \in Q$ as $\pi$ is dense. We will show that the function $i$ is a complete embedding. Since $\{p \in P; \pi(p) \leq q'\} \subseteq \{p \in P; \pi(p) \leq q\}$ whenever $q' \leq q$, it is clear that $i$ is order-preserving. Assume that $i(q) \mid\mid i(q')$ for $q, q' \in Q$; we will show that $q \mid\mid q'$. As we work with a complete Boolean algebra, $i(q) \mid\mid i(q')$ is equivalent to:

$$i(q) \wedge i(q') = \bigvee\{\, p \wedge p' \;;\; \pi(p) \leq q \;\;\&\;\; \pi(p') \leq q' \,\} \neq 0_{\mathrm{RO}^+(P)}. \tag{12}$$

Therefore there are $p$ and $p'$ in $P$ such that $p \wedge p' \neq 0_{\mathrm{RO}^+(P)}$, $\pi(p) \leq q$ and $\pi(p') \leq q'$. By density of $P$ in $\mathrm{RO}^+(P)$, there is $p^* \in P$ below $p \wedge p'$ and as $\pi$ is order-preserving, $\pi(p^*)$ is below both $q$ and $q'$.

To conclude that $i$ is a complete embedding, it suffices by Lemma 3.10 to verify that the image of a maximal antichain is maximal. Let $A$ be a maximal antichain in $Q$, and $p \in P$ be given (it is enough to consider elements of $P$ as $P$ is dense in $\mathrm{RO}^+(P)$). As $A$ is maximal, there is $a \in A$ such that $a$ and $\pi(p)$ are compatible. Hence there is $q \in Q$ which is below $a$ and $\pi(p)$. By (3), there is $p' \leq p$ such that $\pi(p') \leq q$. Since $i(a) = \bigvee\{\, p \in P \;;\; \pi(p) \leq q \,\}$ and $\pi(p') \leq q$, we conclude $p' \leq i(a)$. Therefore the antichain $i\,''A$ is maximal. □

There is a natural method for defining projections from $(P, \leq_P)$ onto suborders of $(\mathrm{RO}^+(Q), \leq_{\mathrm{RO}^+(Q)})$ in situations in which every $P$-generic extension $V[G]$ contains a $Q$-generic filter $H$.

**Lemma 3.12** *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two partial orders. Assume that for every $P$-generic filter $G$ over $V$ there is in $V[G]$ a $Q$-generic filter over $V$. Let $\dot{H}$ be a $P$-name such that $1_P \Vdash$ "$\dot{H}$ is a $\mathrm{RO}^+(Q)$-generic filter".[3] Then the following hold:*

*(i) Define $\pi : P \to \mathrm{RO}^+(Q)$ by*

$$\pi(p) = \bigwedge\{\, b \in \mathrm{RO}^+(Q)\ ;\ p \Vdash b \in \dot{H} \,\}. \tag{13}$$

*Set $b_Q = \pi(1_P) = \bigwedge\{\, b \in \mathrm{RO}^+(Q)\ ;\ 1_P \Vdash b \in \dot{H} \,\}$. Let $\mathrm{RO}^+(Q) \restriction b_Q$ denote the partial order $\{\, b \in \mathrm{RO}^+(Q)\ ;\ b \leq b_Q \,\}$. Then*

$$\pi : P \to \mathrm{RO}^+(Q) \restriction b_Q \text{ is a projection.} \tag{14}$$

*(ii) Moreover, $\pi$ can be defined just using $-Q = \{\, -q\ ;\ q \in Q \,\}$:*

$$\pi(p) = \bigwedge\{\, -q\ ;\ q \in Q\ \&\ p \Vdash -q \in \dot{H} \,\} = \bigwedge\{\, -q\ ;\ q \in Q\ \&\ p \Vdash q \notin \dot{H} \,\}. \tag{15}$$

**Proof** (i). First, we argue that $\pi$ is well defined, i.e. $\pi(p) > 0_{\mathrm{RO}(Q)}$ for all $p \in P$. To see this, denote:

$$H_p = \{\, b \in \mathrm{RO}^+(Q)\ ;\ p \Vdash b \in \dot{H} \,\}. \tag{16}$$

If $\pi(p) = \bigwedge H_p = 0_{\mathrm{RO}^+(Q)}$, then $D = \{\, b \in \mathrm{RO}^+(Q)\ ;\ \exists h \in H_p(h \perp b) \,\}$ is dense. Therefore if $G$ contain $p$, then $H_p \subseteq H = \dot{H}^G$ and also $H \cap D \neq \emptyset$, hence $H$ contains two incompatible elements. This is a contradiction with the assumption that $\dot{H}$ is forced to be an $\mathrm{RO}^+(Q)$-generic filter by $P$.

Notice also that $\pi(p) = \bigwedge H_p$ is forced by $p$ into $\dot{H}$: Consider the following dense set:

$$D = \{\, b \in \mathrm{RO}^+(Q)\ ;\ b \leq \bigwedge H_p\ \vee\ \exists h \in H_p(h \perp b) \,\}. \tag{17}$$

If $G$ contains $p$, but $H$ does not contain $\bigwedge H_p$, then $H$ must meet $D$ in some element incompatible with some element in $H_p$. This is a contradiction. Therefore $p$ forces $\pi(p)$ into $\dot{H}$.

Now, we show that $\pi$ is a projection. The preservation of the ordering is easy. We check condition (3), i.e. for every $p \in P$ and every $c \leq \pi(p)$, there is $p' \leq p$ such that $\pi(p') \leq c$. Let $p$ and $c$ be given. If $c = \pi(p)$, we are trivially done. So suppose $c < \pi(p)$. If for every $p' \leq p$, $p' \nVdash c \in \dot{H}$, then $p \Vdash \pi(p) - c \in \dot{H}$, which contradicts the fact that $\pi(p)$ is the infimum of $H_p = \{\, b \in \mathrm{RO}^+(Q)\ ;\ p \Vdash b \in \dot{H} \,\}$. It follows that there is some $p' \leq p$, $p' \Vdash c \in \dot{H}$. Then $\pi(p') \leq c$ as required.

[3] Notice that $\pi$ defined below depends on the specific name $\dot{H}$ we choose.

(ii). Let $p$ be fixed and let $a_p$ denote $\bigwedge\{ -q ; q \in Q \ \& \ p \Vdash -q \in \dot{H} \}$. We wish to show that $\pi(p)$ from (13) is equal to $a_p$. Clearly $\pi(p) \leq a_p$. For the converse first notice that

$$\pi(p) = \bigwedge\{ -q \, ; \, q \in Q \ \& \ \pi(p) \leq -q \}. \tag{18}$$

This follows from the fact that each element $b$ of $\mathrm{RO}^+(Q)$ can be expressed as a supremum of elements of $Q$ which are below $b$.

Let as denote $\{ -q ; q \in Q \ \& \ \pi(p) \leq -q \}$ by $-Q_p$. To conclude the proof it is enough to show that $-Q_p$ is a subset of $\{ -q \, ; \, q \in Q \ \& \ p \Vdash -q \in \dot{H} \}$, i.e. to prove that if $\pi(p) \leq -q$ then $p \Vdash -q \in \dot{H}$. However, we already proved that $p$ forces $\pi(p)$ into $\dot{H}$, therefore if $-q \geq \pi(p)$ then $p \Vdash -q \in \dot{H}$. □

**Lemma 3.13** *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two partial orders. Assume that for every $P$-generic filter $G$ over $V$, there is in $V[G]$ a $Q$-generic filter over $V$. Let $\dot{H}$ be a $\mathrm{RO}^+(P)$-name such that $1_{\mathrm{RO}^+(P)} \Vdash$ "$\dot{H}$ is a $\mathrm{RO}^+(Q)$-generic filter".[4] Then the following hold:*

*(i) Define $i : \mathrm{RO}^+(Q) \to \mathrm{RO}^+(P)$ by*

$$i(b) = \bigvee\{ a \in \mathrm{RO}^+(P) \, ; \, a \Vdash b \in \dot{H} \}. \tag{19}$$

*Set $b_Q = \bigwedge\{ b \in \mathrm{RO}^+(Q) \, ; \, 1_{\mathrm{RO}^+(P)} \Vdash b \in \dot{H} \}$. Let $\mathrm{RO}^+(Q) \restriction b_Q$ denote the partial order $\{ b \in \mathrm{RO}^+(Q) \, ; \, b \leq b_Q \}$. Then*

$$i : \mathrm{RO}^+(Q) \restriction b_Q \to \mathrm{RO}^+(P) \text{ is a complete embedding,} \tag{20}$$

*where (19) implies $i(b_Q) = 1_{\mathrm{RO}^+(P)}$.*

*(ii) Let $Q \restriction b_Q$ be the partial order $(Q \cap \mathrm{RO}^+(Q) \restriction b_Q) \cup \{b_Q\}$. Then $i' = i \restriction (Q \restriction b_Q)$ from $Q \restriction b_Q$ to $\mathrm{RO}^+(P)$ is a complete embedding.*

*(iii) Moreover, $i'$ can be defined using only the conditions in $P$:*

$$i'(q) = \bigvee\{ p \in P \, ; \, p \Vdash q \in \dot{H} \}. \tag{21}$$

**Proof** (i). First notice that $i$ is well-defined below $b_Q$, i.e. for $b \leq b_Q$ the set $\{ a \in \mathrm{RO}^+(P) ; a \Vdash b \in \dot{H} \}$ is nonempty. Let us denote this set by $\mathrm{RO}^+(P)_b$. If $b = b_Q$, then $i(b) = 1_{\mathrm{RO}^+(P)}$ by the density argument from (17). Assume that $b < b_Q$. If $\mathrm{RO}^+(P)_b$ is empty, then there is no $a \in \mathrm{RO}^+(P)$ with $a \Vdash b \in \dot{H}$, i.e. $1_{\mathrm{RO}^+(P)} \Vdash b \notin \dot{H}$. Then $1_{\mathrm{RO}^+(P)}$ forces $-b \wedge b_Q$ to be in $\dot{H}$ and this is a contradiction as we defined $b_Q$ to be the infimum of the conditions in $\mathrm{RO}^+(Q)$ which are forced into $\dot{H}$ by $1_{\mathrm{RO}^+(P)}$.

Further notice that $i(b)$ forces $b$ into $\dot{H}$. If not, then there is $a$ below $i(b)$ which forces that $b$ is not in $\dot{H}$ but as $a$ is below $i(b) = \bigvee\{ a \in \mathrm{RO}^+(P) \, ; \, a \Vdash b \in \dot{H} \}$, there is $a_0 \leq a$ which forces $b$ into $\dot{H}$. This is a contradiction.

[4] Notice that $i$ defined below depends on the specific name $\dot{H}$ we choose.

If $b \leq b'$, then every $a \in \mathrm{RO}^+(P)$ which forces $b \in \dot{H}$, forces $b'$ in $\dot{H}$ as well, since $\dot{H}$ is forced to be a generic filter, therefore $i$ is order-preserving. The preservation of incompatibility is easy, as compatible conditions cannot force two incompatible conditions into a filter.

To finish the proof, it suffices by Lemma 3.10 to show that the image of a maximal antichan is maximal. Let $A$ be a maximal antichain in $\mathrm{RO}^+(Q)$ and let $b$ in $\mathrm{RO}^+(P)$ be given. As $A$ is a maximal antichain and $\dot{H}$ is forced to be a generic filter, there has to be $a \in A$ and $b' \leq b$ such that $b' \Vdash a \in \dot{H}$. Since $i(a) = \bigvee\{\, b \in \mathrm{RO}^+(P) \,;\, b \Vdash a \in \dot{H} \,\}$, $b' \leq i(a)$ and hence $b \mid\mid i(a)$; therefore $i\,''A$ is maximal.

(ii). This follows from Lemma 3.16(i).

(iii). Let $q$ be fixed and let $a_q$ denote $\bigvee\{\, p \in P \,;\, p \Vdash q \in \dot{H} \,\}$. We show that $i(q)$ as in (19) is equal to $a_q$. Clearly $a_q \leq i(q)$. For the converse, as $i(q)$ is an element of $\mathrm{RO}^+(P)$ and $P$ is dense in $\mathrm{RO}^+(P)$, $i(q) = \bigvee\{\, p \in P \,;\, p \leq i(q) \,\}$; but all conditions below $i(q)$ have to force $q$ in $\dot{H}$, and therefore $i(q) \leq a_q$. □

**Remark 3.14** Note that in the previous two lemmas, Lemma 3.12 and Lemma 3.13, we cannot in general require $\pi(1_P) = 1_{\mathrm{RO}^+(Q)}$ or $i(1_Q) = 1_{\mathrm{RO}^+(P)}$, respectively. Consider the lottery sum of $\mathrm{Add}(\aleph_0, 1)$ and $\mathrm{Add}(\aleph_1, 1)$. It is easy to see that every $\mathrm{Add}(\aleph_0, 1)$-generic filter adds a generic filter for the lottery but only below a condition which chooses $\mathrm{Add}(\aleph_0, 1)$.

We conclude this section by further facts about projections and complete embeddings.

**Lemma 3.15** *Assume $(P, \leq_P)$ and $(Q, \leq_Q)$ are partial orders and $\pi : P \to Q$ is a projection.*

*(i) If $P'$ is dense in $P$, then $\pi \restriction P' : P' \to Q$ is a projection.*

*(ii) (a) If $P$ is dense in $P'$, then there is $\pi' \supseteq \pi$ such that $\pi' : P' \to \mathrm{RO}^+(Q)$ is a projection.*

*(b) If $P'$ is forcing equivalent to $P$, then there is a projection $\pi' : P' \to \mathrm{RO}^+(Q)$.*

*(iii) Let $\dot{R}$ be a $P$-name for a forcing notion. Then $\pi$ naturally extends to a projection $\pi' : P * \dot{R} \to Q$.*

**Proof** (i). Obvious.

(ii)(a). For $p' \in P'$ define

$$\pi'(p') = \bigvee\{\, \pi(p) \,;\, p \in P \;\&\; p \leq p' \,\}. \tag{22}$$

By density of $P$ in $P'$, $\{\, \pi(p) \,;\, p \leq p' \,\}$ is non-empty for every $p'$ and therefore $\pi'(p')$ is in $\mathrm{RO}^+(Q)$ . If $p' \leq q'$ are in $P'$, then clearly $\pi'(p') \leq \pi'(q')$. Suppose $p' \in P'$ is arbitrary and $b \leq \pi'(p')$. By the definition of $\pi'(p')$, there is $b' \leq b$ such that for some $p \leq p'$, $p \in P$, $b' \leq \pi(p)$. It follows there is some $q \leq p \leq p'$, $q \in P$, such that $\pi(q) = \pi'(q) \leq b' \leq b$ as desired.

(ii)(b). As $P$ is dense in $\mathrm{RO}^+(P)$, by the previous item there is a projection $\pi^*$ from $\mathrm{RO}^+(P)$ to $\mathrm{RO}^+(Q)$. Since $P'$ is forcing equivalent to $P$, $P'$ is dense in $\mathrm{RO}^+(P)$, and $\pi' = \pi^* \restriction P'$ is a projection from $P'$ to $\mathrm{RO}^+(Q)$ by (i).

(iii). Define

$$\pi'(p, \dot{r}) = \pi(p), \tag{23}$$

for every $(p, \dot{r})$ in $P * \dot{R}$. If $(p_1, \dot{r}_1) \leq (p_2, \dot{r}_2)$, then in particular $p_1 \leq p_2$, and thus we have $\pi'(p_1, \dot{r}_1) \leq \pi'(p_2, \dot{r}_2)$ because $\pi$ is order-preserving. If $(p, \dot{r})$ is arbitrary and $b \leq \pi'(p, \dot{r}) = \pi(p)$, then since $\pi$ is a projection, there is $p' \leq p$ such that $\pi(p') \leq b$. Since $(p', \dot{r}) \leq (p, \dot{r})$, $\pi'(p', \dot{r}) \leq b$ is as required. □

**Lemma 3.16** *Assume $(P, \leq_P)$ and $(Q, \leq_Q)$ are partial orders and $i : Q \to P$ is a complete embedding.*

*(i) If $Q'$ is dense in $Q$, then $i \restriction Q' : Q' \to P$ is a complete embedding.*

*(ii) (a) If $Q$ is dense in $Q'$, then there is an $i' \supseteq i$ such that $i' : Q' \to \mathrm{RO}^+(P)$ is a complete embedding.*

*(b) If $Q'$ is forcing equivalent to $Q$, then there exists an $i' : Q' \to \mathrm{RO}^+(P)$ which is a complete embedding.*

*(iii) Let $\dot{R}$ be a $P$-name for a forcing notion. Then $i$ naturally extends to a complete embedding $i' : Q \to P * \dot{R}$.*

**Proof** (i). Obvious.

(ii)(a). For $q' \in Q'$ define

$$i'(q') = \bigvee \{\, i(q) \,;\, q \in Q \ \&\ q \leq q' \,\}. \tag{24}$$

By density of $Q$ in $Q'$, $\{\, i(q) \,;\, q \leq q' \,\}$ is non-empty for every $q'$ and therefore $i'(q')$ is in $\mathrm{RO}^+(P)$ . If $q'_0 \leq q'_1$ in $Q'$, then clearly $i'(q'_0) \leq i'(q'_1)$.

Assume that $i'(q'_0)$ is compatible with $i'(q'_1)$, then

$$i'(q'_0) \wedge i'(q'_1) = \bigvee \{ i(q_0) \wedge i(q_1); q_0, q_1 \in Q \ \&\ q_0 \leq q'_0 \ \&\ q_1 \leq q'_1 \} \neq 0_{\mathrm{RO}(P)}. \tag{25}$$

Therefore there are $q_0 \leq q'_0$ and $q_1 \leq q'_1$ such that $i(q_0)$ and $i(q_1)$ are compatible. By the definition of complete embedding, $q_0$ is compatible with $q_1$. Hence $q'_0 \parallel q'_1$, as $q_0 \leq q'_1$ and $q_1 \leq q'_1$.

Suppose $b \in \mathrm{RO}^+(P)$ is arbitrary. Then there is $p \in P$, $p \leq b$, by density of $P$ in $\mathrm{RO}^+(P)$. Therefore there is $q \in Q$ so that for all $q^* \in Q$ such that $q^* \leq q$, $i(q^*)$ is compatible with $p$, hence with $b$. Now, we need to show that for all $q' \in Q'$ such that $q' \leq q$, $i'(q')$ is compatible with $b$. Let $q' \leq q$, $q' \in Q'$, be given and denote $Q_{q'} = \{\, i(q) \,;\, q \in Q \ \&\ q \leq q' \,\}$ so that $i'(q') = \bigvee Q_{q'}$. As all conditions in $Q_{q'}$ are compatible with $b$, and so is $i'(q')$.

(ii)(b). By (a) and the fact that $Q$ is dense in $\mathrm{RO}^+(Q)$ we conclude that there is a complete embedding $i^*$ from $\mathrm{RO}^+(Q)$ to $\mathrm{RO}^+(P)$. Since $Q'$ is forcing equivalent to $Q$,

$Q'$ is dense in $\mathrm{RO}^+(Q)$, hence $i' = i^* \restriction Q'$ is a complete embedding from $Q'$ to $\mathrm{RO}^+(P)$ by (i).

(iii). Define

$$i'(q) = (i(q), 1_{\dot{R}}). \tag{26}$$

If $q_0 \leq q_1$, then $i'(q_0) = (i(q_0), 1_{\dot{R}}) \leq (i(q_1), 1_{\dot{R}}) = i'(q_1)$ because $i$ is order-preserving. The same argument holds for the preservation of incompatibility. Let $(p, \dot{r})$ be arbitrary. Then there is $q \in Q$ such that for all $q' \leq q$, $i(q') \mid\mid p$ and therefore for all $q' \leq q$, $i'(q')$ is compatible with $(p, \dot{r})$. □

## 4 Basic properties of forcing notions

In this section we discuss four basic properties of forcing notions: the chain condition, the Knaster property, closure, and distributivity. We focus on the preservation of these properties by some other forcing notions. Moreover, we state some variations of Easton's lemma which feature more than two forcing notions or deal with distributivity.

**Definition 4.1** Let $P$ be a forcing notion and let $\kappa > \aleph_0$ be a regular cardinal. We say that $P$ is:

- $\kappa$-*cc* if every antichain of $P$ has size less than $\kappa$ (we say that $P$ is *ccc* if it is $\aleph_1$-cc).
- $\kappa$-*Knaster* if for every $X \subseteq P$ with $|X| = \kappa$ there is $Y \subseteq X$, such that $|Y| = \kappa$ and all elements of $Y$ are pairwise compatible.
- $\kappa$-*closed* if every decreasing sequence of conditions in $P$ of size less than $\kappa$ has a lower bound.
- $\kappa$-*distributive* if $P$ does not add new sequences of ordinals of length less than $\kappa$.

It is easy to check that all these properties—except for $\kappa$-closure—are invariant under forcing equivalence. Regarding closure, note that for every non-trivial forcing notion $P$ which is $\kappa$-closed there exists a forcing-equivalent forcing notion which is not even $\aleph_1$-closed (the completion $\mathrm{RO}^+(P)$ is never $\aleph_1$-closed).

**Lemma 4.2** *Let $\kappa > \aleph_0$ be a regular cardinal and assume that $P$ is a forcing notion and $\dot{Q}$ is a $P$-name for a forcing notion. Then the following hold:*

*(i) $P$ is $\kappa$-closed and $P$ forces $\dot{Q}$ is $\kappa$-closed if and only if $P * \dot{Q}$ is $\kappa$-closed.*

*(ii) $P$ is $\kappa$-distributive and $P$ forces $\dot{Q}$ is $\kappa$-distributive if and only if $P * \dot{Q}$ is $\kappa$-distributive.*

*(iii) $P$ is $\kappa$-cc and $P$ forces $\dot{Q}$ is $\kappa$-cc if and only if $P * \dot{Q}$ is $\kappa$-cc.*

**Proof** The proofs are routine; for more details see [Jech03] or [Kun80]. □

An analogous statement (iii) for the Knaster property is not in general true: it may happen that $P * \dot{Q}$ is $\kappa$-Knaster, yet $P$ does not force that $\dot{Q}$ is $\kappa$-Knaster. Consider the following example: Assume $\mathsf{MA}_{\aleph_1}$ and let $\dot{Q}$ be an $\mathrm{Add}(\aleph_0, 1)$-name for the $\aleph_1$-Suslin tree added by $\mathrm{Add}(\aleph_0, 1)$ (see Jech [Jech03] for details). Then $\mathrm{Add}(\aleph_0, 1) * \dot{Q}$ is ccc by previous lemma (iii) and as we assume $\mathsf{MA}_{\aleph_1}$, all ccc forcing notions are $\aleph_1$-Knaster. Therefore $\mathrm{Add}(\aleph_0, 1) * \dot{Q}$ is $\aleph_1$-Knaster, but $\mathrm{Add}(\aleph_0, 1)$ forces that $\dot{Q}$ is not $\aleph_1$-Knaster.

If $Q$ is in the ground model, $P * \check{Q}$ is equivalent to $P \times Q$. Let us state some simple properties of products:

**Lemma 4.3** *Let $\kappa > \aleph_0$ be a regular cardinal and assume that $P$ and $Q$ are forcing notions. Then the following hold:*

*(i) If $P$ and $Q$ are $\kappa$-Knaster, then $P \times Q$ is $\kappa$-Knaster.*

*(ii) If $P$ is $\kappa$-Knaster and $Q$ is $\kappa$-cc, then $P \times Q$ is $\kappa$-cc.*

**Proof** The proofs are routine using only combinatorial arguments (a forcing argument is not required). □

Note that in general Lemma 4.3 cannot be strengthened to say that the product of two $\kappa$-cc forcing notions is $\kappa$-cc (this is called *the productivity of the $\kappa$-cc chain condition*): Consider for instance a Suslin tree $T$ at $\aleph_1$ as a forcing notion; then $T$ is $\aleph_1$-cc, but $T \times T$ has an antichain of size $\aleph_1$. A more complicated example can be constructed under $\mathsf{CH}$; this was first done by Laver in unpublished work, see Galvin [Gal80]. Finally note that $\mathsf{MA}_{\aleph_1}$ implies the $\aleph_1$-cc productivity (in fact, it implies that every $\aleph_1$-cc forcing is $\aleph_1$-Knaster) so there is consistently no such example under $\neg\mathsf{CH}$.

These results are specific to the $\aleph_1$-cc and do not extend to cardinals $\kappa^+ > \aleph_1$: it is provable in $\mathsf{ZFC}$ that for all cardinals $\kappa \geq \aleph_1$, there is a $\kappa^+$-cc forcing whose product is not $\kappa^+$-cc. Examples of such forcings were constructed by Todorcevic and Shelah. The most difficult case of the $\aleph_2$-cc was solved by Shelah in 1997, [She97]. For an overview of productivity of the $\kappa$-chain condition see [Rin14].

The following lemma summarizes some of the more important forcing properties of a product $P \times Q$ regarding the chain condition.

**Lemma 4.4** *Let $\kappa > \aleph_0$ be a regular cardinal and assume that $P$ and $Q$ are forcing notions such that $P$ is $\kappa$-Knaster and $Q$ is $\kappa$-cc. Then the following hold:*

*(i) $P$ forces that $Q$ is $\kappa$-cc.*

*(ii) $Q$ forces that $P$ is $\kappa$-Knaster.*

**Proof** (i). This is an easy consequence of Lemmas 4.2(iii) and 4.3(ii).

(ii). We follow the argument from [Cum18], attributed to Magidor. Let $q \in Q$ be a condition which forces that $\{ \dot{p}_\alpha \,;\, \alpha < \kappa \}$ is a subset of $P$ of size $\kappa$. For each $\alpha$ choose $q_\alpha \leq q$ which decides the value of $\dot{p}_\alpha$ and denote this value by $p_\alpha$. Now, by the $\kappa$-Knasterness of $P$, there is $A \subseteq \kappa$ of size $\kappa$ such that all conditions in $\{ p_\alpha \,;\, \alpha \in A \}$ are pairwise compatible.

Now it suffices to show that there is $q_\alpha$ which forces that $B = \{ \beta \in A \,;\, q_\beta \in \dot{G} \}$ is unbounded in $A$. Then if $G$ is a generic filter containing $q_\alpha$, the set $\{ p_\alpha \,;\, \alpha \in B \}$ is a subset of $\{ \dot{p}_\alpha^G \,;\, \alpha < \kappa \}$ of size $\kappa$ and consists of pairwise compatible conditions.

For contradiction assume that there is no such $\alpha$. It means that for every $\alpha \in A$ we can find $q^*_\alpha \leq q_\alpha$ and $\gamma_\alpha > \alpha$ such that for all $\beta \geq \gamma_\alpha$

$$q^*_\alpha \Vdash q_\beta \notin \dot{G}. \tag{27}$$

In particular $q^*_\alpha$ is incompatible with all $q_\beta$ where $\beta \geq \gamma_\alpha$, and therefore also with all $q^*_\beta$ where $\beta \geq \gamma_\alpha$. Now, it is easy to construct an unbounded subset $A^*$ of $A$ such that all conditions in $\{ q^*_\alpha \,;\, \alpha \in A^* \}$ are pairwise incompatible. This contradicts the assumption that $Q$ is $\kappa$-cc. □

Now we mention some properties of the product with respect to the preservation of $\kappa$-distributivity and $\kappa$-closure. If $P$ and $Q$ are two $\kappa$-distributive forcing notions, then the product $P \times Q$ does not have to be $\kappa$-distributive. Again consider a Suslin tree $T$ at $\aleph_1$ as a forcing notion: $T$ is $\aleph_1$-distributive (see [Jech03] for the details), but $T \times T$ may[5] collapse $\aleph_1$ and therefore it may not be $\aleph_1$-distributive.[6] See also [DJ74] for a construction of a homogeneous $\omega_1$-Suslin tree whose product collapses $\omega_1$, or [JJ74] for a construction of a rigid $\omega_1$-Suslin tree whose product collapses $\omega_1$. For an example in ZFC, consider a stationary and co-stationary subset $S$ of $\omega_1$. Since $S$ and $\omega_1 \setminus S$ are stationary, both forcing notions $\mathrm{CU}(S)$ and $\mathrm{CU}(\omega_1 \setminus S)$ (see Definition 2.6) are $\omega_1$-distributive. Forcing with $\mathrm{CU}(\kappa \setminus S)$ adds a closed unbounded set to $\mathrm{CU}(\kappa \setminus S)$ and hence $S$ is no longer stationary in the generic extension $V[\mathrm{CU}(\kappa \setminus S)]$ and therefore $\mathrm{CU}(S)$ is not distributive in $V[\mathrm{CU}(\kappa \setminus S)]$.

However, if at least one of $P$ and $Q$ is $\kappa$-closed, then the product is $\kappa$-distributive. Moreover, if both $P$ and $Q$ are $\kappa$-closed, then their product is $\kappa$-closed.

The following lemma summarizes some of the important properties of the product $P \times Q$ regarding distributivity and closure.

**Lemma 4.5** *Let $\kappa > \aleph_0$ be a regular cardinal and assume that $P$ and $Q$ are forcing notions, where $P$ is $\kappa$-closed and $Q$ is $\kappa$-distributive. Then the following hold:*

*(i) $P$ forces that $Q$ is $\kappa$-distributive.*

*(ii) $Q$ forces that $P$ is $\kappa$-closed.*

**Proof** The proof is routine. □

We can also formulate some results for the product of two forcing notions with respect to preservation of chain condition and distributivity at the same time. The following lemma appeared in [Eas70].

**Lemma 4.6** (Easton) *Let $\kappa > \aleph_0$ be a regular cardinal and assume that $P$ and $Q$ are forcing notions, where $P$ is $\kappa$-cc and $Q$ is $\kappa$-closed. Then the following hold:*

[5] As we already mentioned, if $T$ is an $\aleph_1$-Suslin tree, then $T \times T$ is not $\aleph_1$-cc, but it can be $\aleph_1$-distributive. An example of such a tree $T$ is the free $\aleph_1$-Suslin tree.

[6] If $P$ is a forcing notion which is $\aleph_1$-distributive, then $P$ does not collapse $\aleph_1$; the converse does not hold in general. However, if $P$ is a tree of height $\omega_1$, then if $P$ does not collapse $\aleph_1$, it must be $\aleph_1$-distributive.

*(i) P forces that Q is $\kappa$-distributive.*

*(ii) Q forces that P is $\kappa$-cc.*

**Proof** For the proof of (i), see [Jech03, Lemma 15.19], (ii) is easy. $\square$

Let us make a few comments regarding the limits of Easton's lemma. We cannot strengthen the conclusion in (i) to $\kappa$-closure: Consider for instance $P = \mathrm{Add}(\aleph_0, 1)$ and $Q = \mathrm{Add}(\aleph_1, 1)$; it is easy to check that $Q$ is not $\aleph_1$-closed in $V[P]$. Similarly, we cannot weaken in general the assumption that $Q$ is $\kappa$-closed to $\kappa$-distributivity: If $T$ is an $\aleph_1$-Suslin tree, then $T$ is $\aleph_1$-distributive and ccc and neither of (i) and (ii) holds for $T$. However, in some cases it suffices to assume that $Q$ is only $\kappa$-distributive:

**Lemma 4.7** *Let $\kappa > \aleph_0$ be a regular cardinal and assume that $P$ and $Q$ are forcing notions, where $P$ is $\kappa$-cc and $Q$ is $\kappa$-distributive. Then if Q forces that $P$ is $\kappa$-cc, then P forces that Q is $\kappa$-distributive.*

**Proof** Let $f$ be a function from some ordinal $< \kappa$ into ordinals in $V[P][Q]$; we want to show that $f$ is in $V[P]$. Note that $V[P][Q] = V[Q][P]$ and since $Q$ forces that $P$ is $\kappa$-cc, $f$ has a nice $P$-name $\dot{f}$ of size $< \kappa$ in $V[Q]$. Since $\dot{f}$ has size $< \kappa$ and $Q$ is $\kappa$-distributive, $\dot{f}$ is already in $V$ and consequently $f$ is in $V[P]$. $\square$

Easton's lemma 4.6 can be generalized in many ways. Let us state one such generalization which combines the chain condition and the closure in a more complicated way (it is probably folklore but we have not found a proof so we give one for the benefit of the reader).

**Lemma 4.8** *Let $\kappa > \aleph_0$ be a regular cardinal, let $P$, $R$, $S$ be forcing notions and let $\dot{Q}$ be a $P$-name for a forcing notion. Assume that $P \times R$ is $\kappa$-cc and P forces that $\dot{Q}$ is $\kappa$-closed. If S is $\kappa$-closed, then $(P * \dot{Q}) \times R$ forces that S is $\kappa$-distributive.*

**Proof** Let use denote $(P * \dot{Q}) \times R$ by $Z$. Assume for simplicity that the greatest condition in $Z \times S$ forces that $\dot{f} : \kappa' \to \mathrm{ORD}$ is a function in $V[Z][S]$ for some fixed $\kappa' < \kappa$ and some name $\dot{f}$. We will find a stronger condition which will force that this function is already in $V[Z]$. As $\dot{f}$ is arbitrary, this will prove the lemma.

By induction in $V$, we construct sequences $w^\alpha = \langle\, ((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta, s^\alpha_\beta) \mid \beta < \gamma_\alpha < \kappa \,\rangle$ for $\alpha < \kappa'$ of conditions in $Z \times S$ with the following properties:

(i) For each $\beta < \gamma_\alpha$, $w^\alpha_\beta = ((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta, s^\alpha_\beta)$ decides the value of $\dot{f}(\alpha)$;

(ii) $1_P$ forces that $\langle\, \dot{q}^\alpha_\beta \mid \beta < \gamma_\alpha \,\rangle$ is a decreasing sequence of conditions in $\dot{Q}$;

(iii) The set $\{\, (p^\alpha_\beta, r^\alpha_\beta)\ ;\ \beta < \gamma_\alpha \,\}$ is a maximal antichain in $P \times R$;

(iv) $\langle\, s^\alpha_\beta \mid \beta < \gamma_\alpha \,\rangle$ forms a decreasing sequence in $S$;

and for $\alpha'$, $\alpha < \alpha' < \kappa'$:

(i) $1_P$ forces that $\dot{q}^{\alpha'}_0$ is below every $\dot{q}^\alpha_\beta$, $\beta < \gamma_\alpha$;

(ii) $s^{\alpha'}_0$ is below every $s^\alpha_\beta$, $\beta < \gamma_\alpha$.

We first construct the sequence $w^0$ by induction, ensuring as we go the conditions (i)–(iv) above. Choose $w^0_0 = ((p^0_0, \dot{q}^0_0), r^0_0, s^0_0)$ so that it decides the value of $\dot{f}(0)$. Suppose $w^0_\beta$ has been constructed for every $\beta < \gamma$; we describe the construction of $w^0_\gamma$. If $\gamma$ is a limit ordinal, first take a lower bound of $\langle\, \dot{q}^0_\beta \mid \beta < \gamma \,\rangle$ (denote it $\dot{q}'$) and a lower bound of $\langle\, s^0_\beta \mid \beta < \gamma \,\rangle$ (denote it $s'$). This is possible by conditions (ii) and (iv), respectively, and from the assumption that $\dot{Q}$ is forced to be $\kappa$-closed and $S$ is $\kappa$-closed. If $\gamma$ is a successor ordinal $\delta + 1$, work with $\dot{q}^0_\delta$ as $\dot{q}'$ and $s^0_\delta$ as $s'$.

If possible, choose a condition $((p, \dot{q}), r, s)$ such that $p$ forces that $\dot{q}$ is below $\dot{q}'$, $s$ is below $s'$, $(p, r)$ is incompatible with all the previous elements $(p^0_\beta, r^0_\beta)$, $\beta < \gamma$, and crucially $((p, \dot{q}), r, s)$ decides the value of $\dot{f}(0)$. In more detail, if possible first pick any $(p', r')$ incompatible with all the previous pairs $(p^0_\beta, r^0_\beta)$, $\beta < \gamma$. Then by the forcing theorem there must be an extension of $((p', \dot{q}'), r', s')$ which decides the value of $\dot{f}(0)$. We denote this extension $((p, \dot{q}), r, s)$ (note that $p \Vdash \dot{q} \leq \dot{q}'$). Set $w^0_\gamma = ((p, \dot{q}''), r, s)$, where $\dot{q}''$ is a name which interprets as $\dot{q}$ below the condition $p$, and interprets as $\dot{q}'$ below conditions incompatible with $p$.

If this is not possible, set $\gamma_0 = \gamma$. Note that $\gamma_0 < \kappa$ since $P \times R$ is $\kappa$-cc.

The construction of $w^\alpha$ for $\alpha < \kappa'$ proceeds analogously, while ensuring the conditions (v)–(vi).

By the $\kappa$-closure of $\dot{Q}$ and $S$, we can take a lower bound of all the conditions appearing in the sequences $w^\alpha$ at the coordinates of $\dot{Q}$ and $S$—denote these lower bounds $\dot{q}$ and $s$, respectively. Let $G \times F$ be any $Z \times S$-generic containing $((1_P, \dot{q}), 1_R, s)$. We want to argue that we can define $\dot{f}^{G\times F}$ already in $V[G]$. Let $\alpha < \kappa$ be fixed. By the construction there is a unique pair $(p^\alpha_\beta, r^\alpha_\beta)$ such that $((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta)$ is in $G$. It follows from the construction of the sequences $w^\alpha$ that $\{\, (p^\alpha_\beta, r^\alpha_\beta) \ ; \ \beta < \gamma_\alpha \,\}$ is a maximal antichain in $P \times R$ by condition (iii). Working in $V[G]$, we can define the right value of $\dot{f}(\alpha)$ as the value which is forced by $((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta, s)$. □

## References


[Abr10] U. Abraham. Proper forcing. In M. Foreman and A. Kanamori, eds., *Handbook of Set Theory*, vol. I, chap. 5, pp. 333–394. Springer, 2010.

[Bau83] J. E. Baumgartner. Iterated forcing. In A. R. D. Mathias, ed., *Surveys in Set Theory*, vol. 87 of *London Math. Soc. Lecture Note Ser.*, pp. 1–59. Cambridge University Press, 1983.

[BHK76] J. E. Baumgartner, L. A. Harrington, and E. M. Kleinberg. Adding a closed unbounded set. *J. Symb. Logic*, 41(2):481–482, 1976.

[Coh63] P. Cohen. The independence of the continuum hypothesis. *Proc. Nat. Acad. Sci. of the U.S.A.*, 50:1143–1148, 1963.

[Coh64] P. Cohen. The independence of the continuum hypothesis II. *Proc. Nat. Acad. Sci. of the U.S.A.*, 51:105–110, 1964.

[Cum18] J. Cummings. Aronszajn and Kurepa trees. *Archive Math. Logic*, 57(1–2):83--90, 2018.

[DJ74] K. Devlin and H. Johnsbråten. *The Souslin Problem*, vol. 405 of *LNM*. Springer, 1974.

[Eas70] W. B. Easton. Powers of regular cardinals. *Ann. Math. Logic*, 1:139–178, 1970.

[FH12] S.-D. Friedman and R. Honzík. Supercompactness and failures of GCH. *Fundamenta Mathematicae*, 219(1):15–36, 2012.

[Gal80] F. Galvin. Chain conditions and products. *Fundamenta Mathematicae*, 108(1):33–48, 1980.

[Ham00] J. D. Hamkins. The lottery preparation. *Ann. Pure Appl. Logic*, 101(2–3):103--146, 2000.

[Jech03] T. Jech. *Set Theory*. Springer Monographs in Mathematics. Springer, 2003.

[Jen] R. Jensen. The generic Kurepa hypothesis I + II. Unpublished handwritten notes.

[JJ74] R. Jensen and H. Johnsbråten. A new construction of a non-constructible $\delta^1_3$ subset of $\omega$. *Fundamenta Mathematicae*, 81:279–290, 1974.

[JS01] T. Jech and S. Shelah. Simple complete Boolean algebras. *Proc. of the Amer. Math. Soc.*, 129(2):543–549, 2001.

[Kan80] A. Kanamori. Perfect-set forcing for uncountable cardinals. *Ann. Math. Logic*, 19:97–114, 1980.

[Kun80] K. Kunen. *Set Theory: An Introduction to Independence Proofs*. North-Holland, 1980.

[Rin14] A. Rinot. Chain conditions of products, and weakly compact cardinals. *Bull. Symb. Logic*, 20(3):293–314, 2014.

[Sac71] G. E. Sacks. Forcing with perfect closed sets. In *Axiomatic Set Theory (Proc. Sympos. Pure Math., Univ. California, Los Angeles CA, 1967)*, vol. XIII, Part I of *Proc. Symp. Pure Math.*, pp. 331–355. Amer. Math. Soc., Providence RI, 1971.

[She97] S. Shelah. Colouring and non-productivity of $\aleph_2$-c.c. *Ann. Pure Appl. Logic*, 84(2):153–174, 1997.

2022 ACTA UNIVERSITATIS CAROLINAE PP. 77–93
**PHILOSOPHICA ET HISTORICA 1** / MISCELLANEA LOGICA

# A NOTE ON GENERALIZED GENERALIZATION

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**ABSTRACT**

The generalization rules of sequent calculi allow, under some restrictions, to derive a formula $\exists x\varphi$ or $\forall x\varphi$ from a formula $\varphi_x(y)$, i.e. from the formula obtained by substituting a variable $y$ for all free occurrences of $x$ in $\varphi$. We introduce modified generalization rules that make it possible to derive $\exists x\varphi$ or $\forall x\varphi$ from $\varphi_x(t)$ even in some cases where $t$ is a complex term. These modified rules were invented in connection with attempts to prove the interpolation theorem for classical predicate logic without equality but with function symbols. This theorem seems (and remains) to be an unresolved case in the literature.

**Keywords:** generalization; sequent; interpolation.

## 1 Introduction: interpolation theorems

Interpolation theorem, for one or another logic, is easily stated if the list of logical symbols includes the "nulary connectives" $\top$ and $\bot$ for truth and falsity. Then the interpolation theorem is the claim that if an implication $\varphi \to \psi$ is *valid* (as determined by the semantics of the logic in question), then there exists a formula $\mu$, called *interpolant* of $\varphi$ and $\psi$, such that $\varphi \to \mu$ and $\mu \to \psi$ are valid and $\mu$ contains only those extralogical symbols that simultaneously occur in both $\varphi$ and $\psi$. In predicate logic we first choose a language $L$, and then *extralogical symbols*, or just *symbols*, are free variables, and function and predicate symbols of $L$. Interpolation makes sense also in various propositional logics (classical, non-classical, modal). Then extralogical symbols are just atoms. We will give some examples of different logics later in Section 3.

Let $\mathrm{Symb}(\varphi)$ or $\mathrm{Symb}(\Gamma)$ for a formula $\varphi$ or a set $\Gamma$ of formulas denote the set of all extralogical symbols in $\varphi$ or in $\Gamma$. Thus an interpolant $\mu$ of formulas $\varphi$ and $\psi$ must satisfy $\mathrm{Symb}(\mu) \subseteq \mathrm{Symb}(\varphi) \cap \mathrm{Symb}(\psi)$. In *propositional logic* $\mathrm{Symb}(..)$ is the set of all atoms in $\varphi$ or in $\Gamma$. In *predicate* logic *with equality* $\mathrm{Symb}(..)$ is the set of all free variables, predicate symbols and function symbols that occur in $\varphi$ or in $\Gamma$. In this case the symbol $=$ has a fixed realization in any structure. It is not considered an extralogical symbol and thus it never appears in a set of the form $\mathrm{Symb}(..)$. Just like connectives and quantifiers, it may occur in an interpolant regardless whether it occurs in the interpolated formulas. In predicate logic *without equality* the symbol $=$ is an extralogical binary symbol with no fixed meaning, and it may occur in an interpolant of $\varphi$ and $\psi$ only if it is in $\mathrm{Symb}(\varphi) \cap \mathrm{Symb}(\psi)$.

https://www.doi.org/10.14712/24647055.2025.5

**Example 1.1** Let $\mathsf{LO}$ be the conjunction of the axioms of strict linear order, i.e. of the sentences $\forall x\forall y\forall z(R(x,y) \,\&\, R(y,z) \to R(x,z))$, $\forall x\forall y(R(x,y) \;\vee\; x = y \;\vee\; R(y,x))$ and $\forall x\neg R(x,x)$. Let $\varphi$ be $\mathsf{LO} \,\&\, \forall x\exists y R(x,y)$ and let $\psi$ be $\exists y(z \neq y)$ where $z \neq y$ is a shorthand for $\neg(z = y)$. In classical predicate logic *with equality* $\mathrm{Symb}(\varphi)$ is $\{R\}$ and $\mathrm{Symb}(\psi)$ is $\{z\}$. The implication $\varphi \to \psi$ is logically valid and thus one can seek an interpolant $\mu$ satisfying $\mathrm{Symb}(\mu) = \emptyset$. It is easy to check that $\mu = \forall x\exists y(x \neq y)$ is as required. In predicate logic *without* equality $\mathrm{Symb}(\varphi)$ is $\{R, =\}$ and $\mathrm{Symb}(\psi)$ is $\{z, =\}$. One can verify that now $\varphi \to \psi$ *is not* logically valid. However, if we denote $\varphi \,\&\, \forall u\forall v(R(u,v) \,\&\, u = v \to R(u,u))$ by $\chi$, then $\chi \to \psi$ is logically valid. Then the same formula $\forall x\exists y(x \neq y)$ works as an interpolant of $\chi$ and $\psi$.

In logic with equality one can use equivalences like $\bot \,\&\, \varphi \equiv \bot$ and $\bot \vee \varphi \equiv \varphi$ and verify that every formula $\mu$ is equivalent to $\top$, or to $\bot$, or to a formula $\nu$ not containing $\top$ and $\bot$ and satisfying $\mathrm{Symb}(\nu) \subseteq \mathrm{Symb}(\mu)$. Thus if $\top$ and $\bot$ are absent, the interpolation theorem reads: if $\varphi \to \psi$ is valid, then $\neg\varphi$ is valid, or $\psi$ is valid, or there exists a formula $\nu$ such that $\varphi \to \nu$ and $\nu \to \psi$ are valid and $\mathrm{Symb}(\nu) \subseteq \mathrm{Symb}(\varphi) \cap \mathrm{Symb}(\psi)$. This explains that we do not see our assumption that $\top$ and $\bot$ are present as a restriction. It just simplifies claims and their proofs. Clearly, $\top$ is equivalent to $\bot \to \bot$ and $\neg\varphi$ is equivalent to $\varphi \to \bot$. Thus when discussing logical calculi, we will be able to simplify their definitions using the assumptions that $\top$ and $\neg$ are defined symbols.

Besides (normal) interpolation one can also consider *uniform interpolation*. Let $\varphi$ be a formula and let $S$ be a set of variables and predicate or function symbols such that $S \subseteq \mathrm{Symb}(\varphi)$. A formula $\mu$ is a *right uniform interpolant of* $\varphi$ *with respect to* $S$ if $\varphi \to \mu$ is valid (again, as determined by the semantics in question), $\mathrm{Symb}(\mu) \subseteq S$, and $\mu \to \psi$ is valid for any formula $\psi$ such that $\mathrm{Symb}(\psi) \cap \mathrm{Symb}(\varphi) \subseteq S$ and $\varphi \to \psi$ is valid. Thus a right uniform interpolant of $\varphi$ with respect to $S$ can be described as the strongest formula $\mu$ that is a consequence of $\varphi$ and satisfies $\mathrm{Symb}(\mu) \subseteq S$. *Left* uniform interpolant is defined analogously.

**Example 1.2** Work in classical predicate logic with equality, let $\mathsf{LO}$ be the same conjunction as in Example 1.1 and let again $\varphi$ be the formula $\mathsf{LO} \,\&\, \forall x\exists y R(x,y)$. We have $\mathrm{Symb}(\varphi) = \{R\}$. Consider a right uniform interpolant $\mu$ of $\varphi$ with respect to $\emptyset$. Then $\mu$ must be a sentence in the language $L_0 = \emptyset$, i.e. a sentence built up from equalities of variables using connectives and quantifiers. Let $m$ be the number of quantifiers in $\mu$. A structure for $L_0$ is just a nonempty set (the structure has a domain and no realizations of symbols). Clearly, every *infinite* structure $\mathcal{A}$ for $L_0$ has an expansion that is a model of $\varphi$. Since $\varphi \to \mu$ is logically valid, we see that $\mu$ is valid in every infinite structure $\mathcal{A}$. However, since $\mu$ contains only $m$ occurrences of quantifiers, it is also valid in every structure having at least $m$ elements. This claim is a consequence of the following lemma, which can be proved by outer induction on $n$ and inner induction on the number of logical symbols in $\psi$. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be structures for* $L_0$, *let* $f : \mathcal{A} \to \mathcal{B}$ *be one-to-one, let* $\psi(x_1, \ldots, x_k)$ *be a formula containing at most* $n$ *quantifiers, let* $a_1, \ldots, a_k$ *be elements of* $\mathcal{A}$, *and assume that* $\mathcal{A}$ *contains at least* $n$ *elements different from* $a_1, \ldots, a_k$ *and* $\mathcal{B}$ *contains at least* $n$ *elements different from* $f(a_1), \ldots, f(a_k)$. *Then* $\mathcal{A} \models \varphi[a_1, \ldots, a_k]$ *if and only if* $\mathcal{B} \models \varphi[f(a_1), \ldots, f(a_k)]$. Knowing that $\mu$ is valid in every structure having at least $m$ elements, we have reached a contradiction: the sentence $\forall x_1 \mathinner{.\,.} \forall x_m \exists y(y \neq x_1 \ \& \ \mathinner{.\,.} \ \& \ y \neq x_m)$ is a consequence of $\varphi$, it is not valid in an $m$-element structure and thus it is not a consequence of $\mu$, which it should be since $\mu$ is a right uniform interpolant. Thus we see that a general theorem stating the existence of right uniform interpolants is not true for classical predicate logic with equality.

First papers about interpolation, containing also some applications, are W. Craig's [Cra57a] and [Cra57b]. Then R. C. Lyndon in [Lyn59a] and [Lyn59b] distinguished positive and negative occurrences of symbols and proved a stronger result: every two formulas $\varphi$ and $\psi$ have an interpolant $\mu$ such that every symbol that appears positively (negatively) in $\mu$ also appears positively (negatively) in both $\varphi$ and $\psi$. Various variants of Craig's or Lyndon's theorem are often cited as the *Craig–Lyndon interpolation theorem*. Henkin in [Hen63] proved (among other things) that uniform interpolation theorems are true for classical propositional logic. Example 1.2 above is also taken from [Hen63]. Later interpolation became a well-established field of research. Now there exists numerous literature about normal or uniform interpolation for different nonclassical logics, and the proofs involve both semantic and proof-theoretic methods. Some idea about this field can be obtained for example from [Bíl07] and from its list of references. Interesting negative results exist as well: [MOU13] show that the interpolation theorem does not hold for logic of constant domains.

Craig and Lyndon proved the interpolation theorem for classical predicate logic with equality, and also for classical predicate logic without equality but with the following additional restriction: *there are no function symbols of nonzero arity*. Also Takeuti and Buss in [Tak75] and [Bus98] work under the same assumption about function symbols. Craig in [Cra57b] says that "most results of this paper do not hold for first-order predicate calculus with function symbols", but does not give any counterexamples. Thus it seems that the case of logic without equality but with no restriction on function symbols is unresolved.

This paper is motivated by this unresolved case, but we will not be able to give an ultimate answer. In the next section we will mention calculi for classical predicate logic. We will put emphasis on Gentzen-style calculi, and we will define generalized (or enhanced) generalization rules that have been invented during attempts to prove an unrestricted interpolation theorem for classical logic without equality. Since we also want to provide the reader with some idea of how the interpolation proofs go, in Section 3 we will survey known proofs for several popular logics. In Section 4 and 5 we will prove that our generalized rules are not sound in logic with equality, but they *are sound* in logic without equality. Thus we perhaps also throw some more light on the role of the equality symbol in logic. The question of unrestricted interpolation theorem for classical logic without equality will remain unanswered.

## 2 Calculi for classical logic, their generalization rules

In Hilbert-style predicate calculi, the generalization rules usually have the following form:

$$\varphi \to \psi \ / \ \exists x \varphi \to \psi \qquad \text{and} \qquad \psi \to \varphi \ / \ \psi \to \forall x \varphi \tag{1}$$

where the variable $x$ has no free occurrences in the formula $\psi$. An advantage of this variant of the generalization rules is that they do not have to be changed when switching to intuitionistic logic. Hilbert-style calculi also have the instantiation axioms and possibly the equality axioms, both being again the same in classical and in intuitionistic logic. The propositional part of a classical Hilbert-style calculus makes it possible to derive every tautology. Here it is good to recall that tautologies are not the same as logically valid formulas: a predicate formula is a tautology if it can be obtained from a propositional tautology by substituting predicate formulas for atoms. As much about Hilbert-style calculi: in the following we will only need Gentzen-style calculi (that is, sequent calculi).

The rules of a *sequent calculus* derive sequents, not formulas. We prefer the definition where sequent is a pair of finite *sets* (rather than multisets or sequences) of formulas. If a sequent consists of sets $\Gamma$ and $\Delta$, we write it as $\langle\, \Gamma \;\Rightarrow\; \Delta \,\rangle$ where $\Rightarrow$ is an auxiliary symbol (not a connective) and the angle brackets just separate the sequent from possible other sequents. Its meaning is "if all formulas in $\Gamma$ hold, then also some formula in $\Delta$ holds". The sets $\Gamma$ and $\Delta$ are called *antecedent* and *succedent* of the sequent $\langle\, \Gamma \;\Rightarrow\; \Delta \,\rangle$. A *rule* of a sequent calculus can be binary (if it derives a sequent from a pair of already proved sequents) or unary (if it derives a sequent from one sequent). A *proof* in a sequent calculus is a tree whose nodes are (labeled by) sequents, every leaf (a node having no predecessors) is an initial sequent and every other sequent is derived from its predecessor or from its two predecessors using a rule. A sequent $\langle\, \Gamma \;\Rightarrow\; \Delta \,\rangle$ is *initial* if $\Gamma \cap \Delta \neq \emptyset$ or if $\bot \in \Gamma$. In fact, initial sequents are nulary rules. A proof is a proof of its root, i.e. of its *endsequent*. A proof of a formula $\varphi$ is a proof of the sequent whose antecedent is empty and whose succedent is $\{\varphi\}$. We write this sequent as $\langle\; \Rightarrow\; \varphi \,\rangle$.

Some rules can be classified as *structural*, i.e. not linked to a logical symbol. The other rules are *logical*. One of the structural rules is *weakening*. It allows adding any formula to antecedent or to succedent. Another structural rule is the *cut rule*, which will be mentioned below. If sequent were defined as a pair of sequences or a pair of multisets, we could also need *contractions* and *exchanges* that make it possible to drop one of two identical formulas or change the order of formulas. Each logical symbol has (logical) rules that "add" a formula in which the symbol occurs at the outermost level. For example, the succedent rules for $\vee$ may look as follows:

$$\frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \vee \psi \,\rangle} \qquad \frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \psi \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \vee \psi \,\rangle} \qquad \frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi, \psi \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \vee \psi \,\rangle.} \tag{2}$$

We follow the usual notation: curly braces enclosing individual formulas are omitted, and commas denote set union, even in expressions like $\mathrm{Symb}(\Gamma, \psi)$. The formula that is "added" by an application of the rule, which in (2) is always $\varphi \vee \psi$, is called *principal formula* of the rule. Once again we have used quotes because the union $\Delta \cup \{\varphi \vee \psi\}$ is legitimate whether $\varphi \vee \psi$ is or is not in $\Delta$, and if it is in $\Delta$, then nothing is added. The formulas that are processed by an application of a rule (the formula $\varphi$, the formula $\psi$ and the two formulas $\varphi$ and $\psi$ in the displayed line (2)) are called *active formulas*. The remaining formulas, which are just copied to the bottom sequent, are *side formulas*.

Given a sequent $\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi, \psi \,\rangle$, one can first apply the first rule in (2) and obtain $\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \vee \psi, \psi \,\rangle$, and then the second rule in (2) yields $\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \vee \psi \,\rangle$.

This reasoning demonstrates that the fact that a principal formula may at the same time be a side formula is very useful, and it also shows that the first two rules, taken together, simulate the third rule. The converse is also true: the third rule can, using a weakening, simulate each of the other two rules.

An example of a binary rule is the antecedent rule for implication. Here we can also opt for one of two variants:

$$\frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \,\rangle \quad \langle\, \Pi, \psi \;\Rightarrow\; \Lambda \,\rangle}{\langle\, \Gamma, \Pi, \varphi \to \psi \;\Rightarrow\; \Delta, \Lambda \,\rangle} \qquad \frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \,\rangle \quad \langle\, \Gamma, \psi \;\Rightarrow\; \Delta \,\rangle}{\langle\, \Gamma, \varphi \to \psi \;\Rightarrow\; \Delta \,\rangle.} \tag{3}$$

In both cases a sequent containing an implication $\varphi \to \psi$ is derived from two sequents, one containing $\varphi$ in the succedent and another containing $\psi$ in the antecedent. The difference is that in the second rule in (3) the two upper sequents have the same sets (the sets $\Gamma$ and $\Delta$) of side formulas. It is the *context-sensitive* variant of the rule, while the first rule, having four sets $\Gamma$, $\Delta$, $\Pi$ and $\Lambda$ of side formulas, is *context-insensitive*. It is clear that the two variants are equivalent (mutually simulable): the context-insensitive variant admits the case where $\Gamma = \Pi$ and $\Delta = \Lambda$, and the context-sensitive variant can simulate the context-insensitive variant with the help of some weakenings.

We do not list the remaining propositional logical rules: the succedent rule for implication, the antecedent rule for disjunction (here one can again opt for a context-sensitive or context-insensitive variant) and the rules for conjunction. The reader may guess (design) them, or they can be found in the literature. Worth mentioning is the *cut rule*:

$$\frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi \,\rangle \quad \langle\, \Pi, \varphi \;\Rightarrow\; \Lambda \,\rangle}{\langle\, \Gamma, \Pi \;\Rightarrow\; \Delta, \Lambda \,\rangle,} \tag{4}$$

which makes it possible to drop a formula if it occurs in the succedent of a proved sequent and in the antecedent of another already proved sequent. A proof not containing an application of the cut rule is a *cut-free proof*. Inspection of the rules other than the cut rule shows that every formula in a cut-free proof is a subformula (in predicate logic, a substitution instance of a subformula) of some formula in the endsequent. Cut-free proofs formalize "direct reasoning", not containing detours through unrelated formulas. Classical logic, both propositional and predicate, satisfies the *cut-elimination theorem*: every provable sequent is provable without using the cut rule. The questions whether the cut-elimination theorem holds, or whether a sequent calculus exists at all, is relevant and studied for every logic.

As to classical logic, we use GK to denote its (more or less just described) calculus. The letters stand for "Gentzen klassisch". In the literature one can also find LK where L refers to "logic". We use the same name GK also for the *predicate* version of the classical calculus, which we will deal with now. The generalization rules of GK are

$$\frac{\langle\, \Gamma, \varphi_x(y) \;\Rightarrow\; \Delta \,\rangle}{\langle\, \Gamma, \exists x \varphi \;\Rightarrow\; \Delta \,\rangle} \qquad \text{and} \qquad \frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi_x(y) \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \forall x \varphi \,\rangle} \tag{5}$$

where $\varphi_x(y)$ denotes the result of substituting $y$ for all free occurrences of the variable $x$ in $\varphi$, and $y$, the *eigenvariable*, is a variable substitutable for $x$ in $\varphi$ that has no free occurrences in the resulting sequent $\langle\, \Gamma, \exists x\varphi \;\Rightarrow\; \Delta \,\rangle$ or $\langle\, \Gamma \;\Rightarrow\; \Delta, \forall x\varphi \,\rangle$. Thus Hilbert-style calculi and sequent calculi share a restriction concerning the variable that is generalized. The rules (5), furthermore, make it possible to rename this variable. This difference is not essential: while the rules (1) do not allow renaming, in a Hilbert-style calculus renaming of bound variables can, of course, be achieved. The remaining quantifier rules of GK are the instantiation (or specification) rules:

$$\frac{\langle\, \Gamma, \varphi_x(t) \;\Rightarrow\; \Delta \,\rangle}{\langle\, \Gamma, \forall x\varphi \;\Rightarrow\; \Delta \,\rangle} \qquad \text{and} \qquad \frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi_x(t) \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \exists x\varphi \,\rangle} \tag{6}$$

where, again, $\varphi_x(t)$ denotes the result of substituting $t$ for all free occurrences of $x$ in $\varphi$, and $t$ is a term of the language in question that is substitutable for $x$ in $\varphi$. It is good to notice the common properties and the differences between the generalization and instantiation rules. In both (5) and (6) a quantified formula is obtained by "unsubstituting" a substitutable term. However, in (5) this term must be a variable and it must not occur in the resulting sequent. The latter stipulation is called *eigenvariable condition*, and it is easy to verify that without it the rules (5) would not be sound with respect to the classical (i.e. Tarskian) semantics.

The generalization rules correspond to reasoning that appears in virtually every mathematical proof. For example, the second rule in (5) formalizes the following argument.

> We have to show that every individual has the property $\varphi$. Let an individual $y$ be given. [...]. Therefore, $y$ has the property $\varphi$. Since $y$ was arbitrary, all individuals have the property $\varphi$.

This reasoning is sound if $y$ is a *new* variable, i.e. if $y$ does not denote anything else in the proof in question. And this is exactly the stipulation to which the eigenvariable condition corresponds. The first rule in (5) corresponds to a logical step that frequently occurs as well. This is not a surprise since in classical logic the quantifiers $\exists$ and $\forall$ behave symmetrically and are interdefinable.

In this paper we consider the following *enhanced*, or generalized, *generalization rules*:

$$\frac{\langle\, \Gamma, \varphi_{x_1,..,x_n}(t_1,..,t_n) \;\Rightarrow\; \Delta \,\rangle}{\langle\, \Gamma, \exists x_1 .. \exists x_n\varphi \;\Rightarrow\; \Delta \,\rangle} \qquad \text{and} \qquad \frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi_{x_1,..,x_n}(t_1,..,t_n) \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \forall x_1 .. \forall x_n\varphi \,\rangle} \tag{7}$$

where $t_1, .., t_n$ are pairwise different terms that are substitutable for $x_1, .., x_n$ in $\varphi$ and such that, for each $i$, the outermost function symbol of $t_i$ (the term $t_i$ itself if it is a variable) has no occurrences (has no free occurrences) in the resulting sequent (in the bottom). The terms $t_i$ can contain inner occurrences of arbitrary function symbols and of arbitrary variables. We cannot claim that these enhanced rules correspond to some logical steps in real proofs. Indeed, we never write something so strange like this:

> We have to show that every individual $x$ is in the relation $\varphi$ to $z$, i.e. that it satisfies $\varphi(x, z)$. Let an individual be given and let us denote it by $G(z)$. [...]. Since $G(z)$ is in the relation $\varphi$ to $z$, we indeed have $\forall x\varphi(x, z)$.

However, modified generalization rules like (7) can be useful when thinking about interpolation in predicate logic and about its proof-theoretic proofs.

## 3 Aspects of interpolation proofs

In the literature there exist both semantic and proof-theoretic proofs of interpolation theorems. In this section we will survey known proof-theoretic proofs for several popular logics. We will also reproduce a proof (known from various sources like [Tak75] and [Bus98]) for classical predicate logic without function symbols of nonzero arity.

A proof-theoretic proof of an interpolation theorem usually consists in two steps: first finding a sequent form of the theorem, i.e. formulating a claim concerning provable sequents, and then proving that claim by induction on the depth of a cut-free proof $\mathcal{P}$. The steps presuppose that the completeness theorem and the cut-elimination theorem hold for the given logic. In the case of *classical propositional logic*, where we know that a formula $\varphi$ is a tautology if and only if the sequent $\langle\ \Rightarrow\ \varphi\,\rangle$ is provable in GK and that the cut-elimination theorem is true for GK, the claim can be as follows. *Let $\mathcal{P}$ be a cut-free proof of $\langle\,\Gamma;\Pi\ \Rightarrow\ \Delta;\Lambda\,\rangle$. Then there exists a formula $\mu$ such that* $\mathrm{Symb}(\mu)\subseteq(\Gamma\cup\Delta)\cap(\Pi\cup\Lambda)$ *and both* $\langle\,\Gamma\ \Rightarrow\ \Delta,\mu\,\rangle$ *and* $\langle\,\Pi,\mu\ \Rightarrow\ \Lambda\,\rangle$ *are provable.* The semicolons denote set union just like commas, but in addition they indicate how the given sequent is divided into two sequents $\langle\,\Gamma\ \Rightarrow\ \Delta\,\rangle$ and $\langle\,\Pi\ \Rightarrow\ \Lambda\,\rangle$. The sets $\Gamma$ and $\Pi$ and also the sets $\Delta$ and $\Lambda$ do not have to be disjoint. Once this claim is proved, the interpolation theorem follows: if $\varphi\to\psi$ is a tautology, then $\langle\,\varphi;\ \Rightarrow;\psi\,\rangle$ is provable, and then a formula $\mu$ obtained by the claim is an interpolant of $\varphi$ and $\psi$.

If an initial sequent, i.e. an endsequent of a zero-depth proof, is divided into two sequents, we have one of the following six situations. Recall the agreement that $\bot$ is a basic symbol and that $\top$ and $\neg\varphi$ are considered shorthands for $\bot\to\bot$ and $\varphi\to\bot$:

$$\begin{array}{ll}\langle\,\Gamma,\bot;\Pi\ \Rightarrow\ \Delta;\Lambda\,\rangle, & \langle\,\Gamma;\bot,\Pi\ \Rightarrow\ \Delta;\Lambda\,\rangle\\ \langle\,\Gamma,\varphi;\Pi\ \Rightarrow\ \Delta;\varphi,\Lambda\,\rangle, & \langle\,\Gamma;\varphi,\Pi\ \Rightarrow\ \Delta;\varphi,\Lambda\,\rangle\\ \langle\,\Gamma,\varphi;\Pi\ \Rightarrow\ \Delta,\varphi;\Lambda\,\rangle, & \langle\,\Gamma;\varphi,\Pi\ \Rightarrow\ \Delta,\varphi;\Lambda\,\rangle.\end{array}\tag{8}$$

One can easily check that the six formulas $\bot$, $\bot\to\bot$, $\varphi$, $\bot\to\bot$, $\bot$ and $\varphi\to\bot$, respectively, satisfy the requirements on interpolant. For example in the third case every extralogical symbol in $\varphi$ occurs in both $\mathrm{Symb}(\Gamma,\varphi,\Delta)$ and $\mathrm{Symb}(\Pi,\varphi,\Lambda)$ and both $\langle\,\Gamma,\varphi\ \Rightarrow\ \Delta,\varphi\,\rangle$ where $\varphi$ is added to the succedent, and $\langle\,\Pi,\varphi\ \Rightarrow\ \varphi,\Lambda\,\rangle$ where $\varphi$ is added to the antecedent, are provable. In the first case both $\langle\,\Gamma,\bot\ \Rightarrow\ \Delta,\bot\,\rangle$ and $\langle\,\Pi,\bot\ \Rightarrow\ \Lambda\,\rangle$ are provable, and the stipulation concerning symbols is satisfied because $\mathrm{Symb}(\bot)=\emptyset$. Notice also that the last case would be problematic in intuitionistic logic. The provability of $\langle\,\Gamma\ \Rightarrow\ \Delta,\varphi,\varphi\to\bot\,\rangle$ is based on the provability of $\langle\ \Rightarrow\ \varphi,\varphi\to\bot\,\rangle$, and the latter sequent is in fact the same as the disjunction $\varphi\vee\neg\varphi$.

We proceed to the induction step. Let a nonzero-depth cut-free proof $\mathcal{P}$ of a sequent divided by semicolons into two subsequents be given. Distinguish the cases whether the last inference in $\mathcal{P}$ is an application of one or another rule and whether the principal formula of that inference is before or after a semicolon. For example, if the last inference of $\mathcal{P}$ is the antecedent $\to$-rule and its principal formula $\varphi\to\psi$ is after the semicolon, we have:

$$\frac{\langle\,\Gamma;\Pi\ \Rightarrow\ \Delta;\varphi,\Lambda\,\rangle\qquad\langle\,\Gamma;\psi,\Pi\ \Rightarrow\ \Delta;\Lambda\,\rangle}{\langle\,\Gamma;\varphi\to\psi,\Pi\ \Rightarrow\ \Delta;\Lambda\,\rangle.}\tag{9}$$

We for simplicity assume that the binary rules of our calculus are context-sensitive. The depths of the subproofs $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\langle\, \Gamma; \Pi \;\Rightarrow\; \Delta; \varphi, \Lambda \,\rangle$ and $\langle\, \Gamma; \psi, \Pi \;\Rightarrow\; \Delta; \Lambda \,\rangle$ are less than the depth of $\mathcal{P}$ and thus the induction hypothesis is applicable. It says that if we arbitrarily divide the endsequents of $\mathcal{P}_1$ and $\mathcal{P}_2$ into subsequents, then a required formula exists. We do not have to be creative when dividing the two endsequents: since it is given that $\varphi \to \psi$ is after the semicolon, in the upper sequents we just put the semicolons before the active formulas $\varphi$ and $\psi$. Let $\varepsilon$ and $\nu$ be interpolants of the endsequents of $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Thus the following four sequents are provable:

$$\begin{aligned} &\langle\, \Gamma \;\Rightarrow\; \Delta, \varepsilon \,\rangle, \qquad \langle\, \Pi, \varepsilon \;\Rightarrow\; \varphi, \Lambda \,\rangle, \\ &\langle\, \Gamma \;\Rightarrow\; \Delta, \nu \,\rangle, \qquad \langle\, \psi, \Pi, \nu \;\Rightarrow\; \Lambda \,\rangle. \end{aligned} \tag{10}$$

We have not written down the succedent &-rule, but it is natural and makes it possible to derive $\langle\, \Gamma \;\Rightarrow\; \Delta, \varepsilon \,\&\, \nu \,\rangle$ from the first and third sequents. Using the antecedent &-rule, the sequents $\langle\, \Pi, \varepsilon \,\&\, \nu \;\Rightarrow\; \varphi, \Lambda \,\rangle$ and $\langle\, \psi, \Pi, \varepsilon \,\&\, \nu \;\Rightarrow\; \Lambda \,\rangle$ can be obtained from the second and fourth sequent respectively, and they yield $\langle\, \Pi, \varphi \to \psi, \varepsilon \,\&\, \nu \;\Rightarrow\; \Lambda \,\rangle$ using the antecedent implication rule. Since every atom in $\varepsilon$ is in both $\mathrm{Symb}(\Gamma, \Delta)$ and $\mathrm{Symb}(\Pi, \varphi, \Lambda)$, and every atom in $\nu$ is in both $\mathrm{Symb}(\Gamma, \Delta)$ and $\mathrm{Symb}(\psi, \Pi, \Lambda)$, it is clear that the formula $\varepsilon \,\&\, \nu$ is built up only from atoms that occur in both $\mathrm{Symb}(\Gamma, \Delta)$ and $\mathrm{Symb}(\Pi, \varphi \to \psi, \Lambda)$. We see that the conjunction $\varepsilon \,\&\, \nu$ satisfies all requirements, and thus it is an interpolant of $\langle\, \Gamma; \varphi \to \psi, \Pi \;\Rightarrow\; \Delta; \Lambda \,\rangle$.

All other cases are treated similarly. In the case of a binary rule, the conjunction or the disjunction of the interpolants of the upper sequents always works as an interpolant of the endsequent of the whole proof $\mathcal{P}$. In the case of a unary rule an interpolant of the upper sequent satisfies the requirements for an interpolant of the endsequent.

In the definition of the calculus GK one can insist that the principal formulas of initial sequents be atomic. From this fact one can obtain a somewhat stronger version of the interpolation theorem for classical propositional logic: for any two formulas $\varphi$ and $\psi$ such that $\varphi \to \psi$ is a tautology there exists an interpolant built up from atoms and negated atoms using conjunctions and disjunctions only.

In *modal logic* we have an additional unary logical symbol $\Box$. A formula $\Box\varphi$ is read "necessarily $\varphi$". Besides $\Box$, the necessity operator, one can also consider $\Diamond$, the possibility operator. However, it is usually considered a defined symbol: $\Diamond\varphi$ is a shorthand for $\neg\Box\neg\varphi$. One of extensively studied propositional modal logics is *provability logic*. Different symbolic names for this logic can be found in the literature. Now, after about fifty years history, it is usually denoted by GL where the letters refer to Gödel and Löb. The semantics (one of semantics) for GL is based on the idea to understand the $\Box$ operator (interpret it, translate it to) *provability* in some recursively axiomatized and sufficiently strong axiomatic theory, *formalized* in the same (or sometimes different) axiomatic theory. In GL one can model reasoning about self-referential sentences, and GL also has some applications in this field and thus in meta-mathematics. One of these applications is that, under some circumstances, a sentences defined by self-reference is unique up to provable equivalence.

Hilbert-style calculus for provability logic is based on the axioms K and 4 that traditionally occur in say more philosophically oriented literature, and on the Löb's axiom schema $\Box(\Box\varphi \to \varphi) \to \Box\varphi$. Sequent calculus for GL was defined in [SV82]. It is based on a single modal rule, which is sufficient to simulate the axioms K and 4 as well as the Löb's axiom:

$$\frac{\langle\, \Gamma, \Box\Gamma, \Box\varphi \;\Rightarrow\; \varphi \,\rangle}{\langle\, \Box\Gamma \;\Rightarrow\; \Box\varphi \,\rangle.} \tag{11}$$

Here $\Box\Gamma$ denotes the set $\{\, \Box\psi \;;\; \psi \in \Gamma \,\}$. The rule is applicable on a sequent $\mathcal{S}$ only if (i) the succedent of $\mathcal{S}$ consists of *exactly one* formula $\varphi$, (ii) the antecedent of $\mathcal{S}$ contains $\Box\varphi$, and (iii) the rest of the antecedent consists of pairs $\psi$ and $\Box\psi$. The conditions (ii) and (iii) are not really demanding because one can always add some formulas using the weakening rule. All formulas in the bottom sequent of (11) begin with $\Box$.

The sequent calculus for GL satisfies cut-elimination, and the interpolation theorem for GL can be proved along the same lines as for classical propositional logic. That is, we prove the same claim concerning a sequent $\langle\, \Gamma; \Pi \;\Rightarrow\; \Delta, \Lambda \,\rangle$ by induction on the depth of its cut-free proof $\mathcal{P}$. Most cases are the same as above, but there are two additional cases to consider: if the last inference of $\mathcal{P}$ is an application of the modal rule and its principal formula $\Box\varphi$ occurs before, or after the semicolon. Let us discuss the former case, the latter is treated similarly. The endsequent of $\mathcal{P}$ thus has the form $\langle\, \Box\Gamma; \Box\Pi \;\Rightarrow\; \Box\varphi; \,\rangle$. The sequent to which the modal rule is applied must have $\Gamma$, $\Box\Gamma$, $\Pi$, $\Box\Pi$ and $\Box\varphi$ in the antecedent and $\varphi$ in the succedent, and we apply the induction hypothesis (in the expected way) on the sequent $\langle\, \Gamma, \Box\Gamma, \Box\varphi; \Pi, \Box\Pi \;\Rightarrow\; \varphi; \,\rangle$. Thus there exists a formula $\nu$ and proofs $\mathcal{P}_1$ and $\mathcal{P}_2$ of the sequents $\langle\, \Gamma, \Box\Gamma, \Box\varphi \;\Rightarrow\; \varphi, \nu \,\rangle$ and $\langle\, \Pi, \Box\Pi, \nu \;\Rightarrow\; \,\rangle$. The proofs $\mathcal{P}_1$ and $\mathcal{P}_2$ can be extended as follows:

$$\cfrac{\cfrac{\cfrac{\langle\, \Gamma, \Box\Gamma, \Box\varphi \;\Rightarrow\; \varphi, \nu \,\rangle}{\langle\, \Gamma, \Box\Gamma, \Box\varphi, \neg\nu, \Box\neg\nu \;\Rightarrow\; \varphi \,\rangle}}{\langle\, \Box\Gamma, \Box\neg\nu \;\Rightarrow\; \Box\varphi \,\rangle}}{\langle\, \Box\Gamma \;\Rightarrow\; \Box\varphi, \neg\Box\neg\nu \,\rangle} \qquad \cfrac{\cfrac{\cfrac{\langle\, \Pi, \Box\Pi, \nu \;\Rightarrow\; \,\rangle}{\langle\, \Pi, \Box\Pi, \Box\neg\nu \;\Rightarrow\; \neg\nu \,\rangle}}{\langle\, \Box\Pi \;\Rightarrow\; \Box\neg\nu \,\rangle}}{\langle\, \Box\Pi, \neg\Box\neg\nu \;\Rightarrow\; \,\rangle}$$

In the left we have first negated $\nu$ and moved it to the other side of the sequent. This is exactly what the $\neg$-rules do. We have also added the formula $\Box\neg\nu$ via the weakening rule, and we did it in one line to save space. Then the modal rule is applicable, and finally the endsequent is obtained by another application of the $\neg$-rule. The explanation for the proof in the right is similar. Since every symbol (every atom) in $\nu$ is in both $\mathrm{Symb}(\Gamma, \Box\Gamma, \Box\varphi, \varphi)$ and $\mathrm{Symb}(\Pi, \Box\Pi)$, it is clear that every atom in $\neg\Box\neg\nu$ is in both $\mathrm{Symb}(\Box\Gamma, \Box\varphi)$ and $\mathrm{Symb}(\Box\Pi)$. Thus $\mu = \neg\Box\neg\nu$ is as required.

The above formal proofs can be easily modified for the case where $\neg$ is not considered a basic symbol. However, the presence of $\bot$ is essential in GL. Without it, the formulas $\varphi = \Box(p \mathbin{\&} \neg p)$ and $\psi = \Box(q \mathbin{\&} \neg q)$, of which $\varphi$ is not refutable and $\psi$ is not provable in GL, would have no interpolant.

A possible exercise could be this: take $\varphi = \neg\Box p$ and $\psi = \Box(q \to \neg\Box q) \to \neg\Box q$, prove $\varphi \to \psi$ in the sequent calculus and find an interpolant of these two formulas. The choice of $\psi$ is motivated by Gödel's first incompleteness theorem: if a sentence $q$ is provably equivalent to its own unprovability, or, if it just *implies* its own uprovability, then it is unprovable.

Provability logic has a satisfactory Kripke semantics. Its Hilbert-style and sequent calculi polynomially simulate each other and are complete with respect to transitive reversely well-founded trees, and also with respect to (the smaller class of all) finite transitive and irreflexive trees. To show completeness of the sequent calculus with respect to Kripke semantics, one can prove (Sambin and Valentini in [SV82] prove) the following claim: every sequent either has a Kripke counterexample, or a cut-free proof. This way the completeness and the cut-elimination theorem are proved at the same time. Similarly, i.e. via a semantic detour, one can actually prove the cut-elimination theorem for each logic mentioned in this paper. A direct proof for GL, i.e. an algorithm that, given a proof, outputs a cut-free proof of the same sequent, was published in [GR08]. GL is also complete with respect to the arithmetic semantics. This is a famous Solovay's result published in [Sol76].

A sequent calculus for *intuitionistic logic* can be obtained by the following modification of GK: the succedent rules for $\neg$, $\rightarrow$ and $\forall$ do not admit side formulas in succedent. Thus after one of these rules is used, the succedent is a singleton consisting of the principal formula. We call this calculus GJ, where G again refers to Gentzen. Many authors (like Takeuti in [Tak75]) use LJ to denote this calculus. A related calculus $\mathrm{GJ}^1$ is based on an even stronger restriction: each succedent in a $\mathrm{GJ}^1$-proof must be empty or a singleton. Our assumption that $\neg$ is a defined symbol again simplifies matters, and it also has the following consequence: each succedent in a $\mathrm{GJ}^1$-proof contains *exactly one* formula. This is so because no rule except the $\neg$-rules can change the number of formulas in succedent. Thus the $\rightarrow$-rules of $\mathrm{GJ}^1$ are

$$\frac{\langle\, \Gamma \;\Rightarrow\; \varphi \,\rangle \qquad \langle\, \Gamma, \psi \;\Rightarrow\; \delta \,\rangle}{\langle\, \Gamma, \varphi \rightarrow \psi \;\Rightarrow\; \delta \,\rangle} \qquad\qquad \frac{\langle\, \Gamma, \varphi \;\Rightarrow\; \psi \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \varphi \rightarrow \psi \,\rangle}$$

where the rule in the right, the succedent implication rule, is the same as in GJ. The &-rules of $\mathrm{GJ}^1$ are:

$$\frac{\langle\, \Gamma, \varphi \;\Rightarrow\; \delta \,\rangle}{\langle\, \Gamma, \varphi \,\&\, \psi \;\Rightarrow\; \delta \,\rangle} \qquad \frac{\langle\, \Gamma, \psi \;\Rightarrow\; \delta \,\rangle}{\langle\, \Gamma, \varphi \,\&\, \psi \;\Rightarrow\; \delta \,\rangle} \qquad \frac{\langle\, \Gamma \;\Rightarrow\; \varphi \,\rangle \qquad \langle\, \Gamma \;\Rightarrow\; \psi \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \varphi \,\&\, \psi \,\rangle.}$$

The completeness and cut-elimination theorems hold for both GJ and $\mathrm{GJ}^1$. It is not clear (to the present author) how about GJ, but $\mathrm{GJ}^1$ can be used to prove the interpolation theorem for intuitionistic propositional logic. We follow the proof in [Min02].

As in other cases, we proceed by induction on the depth of a cut-free proof. However, the claim we prove is now different: *for any cut-free proof of a sequent* $\langle\, \Gamma; \Pi \;\Rightarrow\; \lambda \,\rangle$ *in the calculus* $GJ^1$ *there exists a formula* $\mu$ *such that* $\langle\, \Gamma \;\Rightarrow\; \mu \,\rangle$ *and* $\langle\, \Pi, \mu \;\Rightarrow\; \lambda \,\rangle$ *are provable and all atoms in* $\mu$ *are in both* $\mathrm{Symb}(\Gamma)$ *and* $\mathrm{Symb}(\Pi, \lambda)$. Now there are no semicolons in succedents. To prove this claim is sufficient for our goal: to find an interpolant of a pair $\varphi$ and $\psi$, it is enough to put $\Gamma = \{\varphi\}$, $\Pi = \emptyset$ and $\lambda = \psi$. The base case is as follows. If $\langle\, \Gamma; \Pi \;\Rightarrow\; \lambda \,\rangle$ is an initial sequent (the endsequent of a zero-depth proof $\mathcal{P}$), we deal with the following four cases:

$$\langle\, \Gamma, \bot; \Pi \;\Rightarrow\; \lambda \,\rangle, \quad \langle\, \Gamma; \bot, \Pi \;\Rightarrow\; \lambda \,\rangle, \quad \langle\, \Gamma, \varphi; \Pi \;\Rightarrow\; \lambda \,\rangle, \quad \langle\, \Gamma; \varphi, \Pi \;\Rightarrow\; \lambda \,\rangle$$

and it is straightforward to verify that $\bot$, $\bot \rightarrow \bot$, $\varphi$ and $\bot \rightarrow \bot$ can be picked for $\mu$.

As to the induction step, most cases are the same as in classical logic. For example, if the last inference in a proof $\mathcal{P}$ derives $\langle\, \Gamma; \varphi \to \psi, \Pi \;\Rightarrow\; \lambda \,\rangle$ from a sequent whose succedent is $\{\varphi\}$ and from another sequent having $\psi$ in the antecedent, we write the two sequents as $\langle\, \Gamma; \Pi \;\Rightarrow\; \varphi \,\rangle$ and $\langle\, \Gamma; \psi, \Pi \;\Rightarrow\; \lambda \,\rangle$. Then the induction step yields formulas $\varepsilon$ and $\nu$ such that all atoms in $\varepsilon$ are in both $\mathrm{Symb}(\Gamma)$ and $\mathrm{Symb}(\Pi, \varphi)$ all atoms in $\nu$ are in both $\mathrm{Symb}(\Gamma)$ and $\mathrm{Symb}(\psi, \Pi, \lambda)$ and the sequents $\langle\, \Gamma \;\Rightarrow\; \varepsilon \,\rangle$, $\langle\, \Pi, \varepsilon \;\Rightarrow\; \varphi \,\rangle$, $\langle\, \Gamma \;\Rightarrow\; \nu \,\rangle$ and $\langle\, \psi, \Pi, \nu \;\Rightarrow\; \lambda \,\rangle$ are provable. Then it is easy to verify that $\mu = \varepsilon \,\&\, \nu$ is as required.

A case that cannot be simply copied from classical logic is when the endsequent of $\mathcal{P}$ is given as $\langle\, \Gamma, \varphi \to \psi; \Pi \;\Rightarrow\; \lambda \,\rangle$ with $\varphi \to \psi$ being a principal formula. In this case we use the right to divide the two preceding sequents as needed, and we write them as $\langle\, \Pi; \Gamma \;\Rightarrow\; \varphi \,\rangle$ and $\langle\, \Gamma, \psi; \Pi \;\Rightarrow\; \lambda \,\rangle$. The induction hypothesis yields formulas $\varepsilon$ and $\nu$ such that the sequents

$$\langle\, \Pi \;\Rightarrow\; \varepsilon \,\rangle, \quad \langle\, \Gamma, \varepsilon \;\Rightarrow\; \varphi \,\rangle, \quad \langle\, \Gamma, \psi \;\Rightarrow\; \nu \,\rangle, \quad \langle\, \Pi, \nu \;\Rightarrow\; \lambda \,\rangle$$

are provable. Then from the second and third, and from the first and fourth of them we can continue as follows:

$$\cfrac{\cfrac{\langle\, \Gamma, \varepsilon \;\Rightarrow\; \varphi \,\rangle \qquad \langle\, \Gamma, \psi \;\Rightarrow\; \nu \,\rangle}{\langle\, \Gamma, \varphi \to \psi, \varepsilon \;\Rightarrow\; \nu \,\rangle}}{\langle\, \Gamma, \varphi \to \psi \;\Rightarrow\; \varepsilon \to \nu \,\rangle} \qquad\qquad \cfrac{\langle\, \Pi \;\Rightarrow\; \varepsilon \,\rangle \qquad \langle\, \Pi, \nu \;\Rightarrow\; \lambda \,\rangle}{\langle\, \Pi, \varepsilon \to \nu \;\Rightarrow\; \lambda \,\rangle}$$

Since all atoms in $\varepsilon$ are in both $\mathrm{Symb}(\Pi)$ and $\mathrm{Symb}(\Gamma, \varphi)$ and all atoms in $\nu$ are in both $\mathrm{Symb}(\Gamma, \psi)$ and $\mathrm{Symb}(\Pi, \lambda)$, we see that all atoms in $\varepsilon \to \nu$ are in both $\mathrm{Symb}(\Gamma, \varphi \to \psi)$ and $\mathrm{Symb}(\Pi, \lambda)$. Thus $\mu = \varepsilon \to \nu$ is as required.

In *classical predicate logic without equality* we can stick with the same claim as in classical propositional logic, but we have to consider the generalization rules (5) and the instantiation rules (6). Generalization poses no problem. Indeed, let a proof $\mathcal{P}$ of a sequent divided by semicolons be given, let its last inference be an application of the antecedent $\exists$-rule with a principal formula $\exists x \varphi$:

$$\frac{\langle\, \Gamma; \varphi_x(y), \Delta \;\Rightarrow\; \Pi; \Lambda \,\rangle}{\langle\, \Gamma; \exists x \varphi, \Delta \;\Rightarrow\; \Pi; \Lambda \,\rangle} \tag{12}$$

and let $\nu$ be such that $\langle\, \Gamma \;\Rightarrow\; \Pi, \nu \,\rangle$ and $\langle\, \varphi_x(y), \Delta, \nu \;\Rightarrow\; \Lambda \,\rangle$ are provable and $\mathrm{Symb}(\nu)$ is a subset of both $\mathrm{Symb}(\Gamma, \Pi)$ and $\mathrm{Symb}(\varphi_x(y), \Delta, \Lambda)$. From the fact that the variable $y$ satisfies the eigenvariable condition we can draw several consequences. (i) Since $y$ is not free in formulas in $\Gamma$ and $\Pi$, from $\mathrm{Symb}(\nu) \subseteq \mathrm{Symb}(\Gamma, \Pi)$ it is clear that $y$ is not free in $\nu$. (ii) Once we know that, from $\mathrm{Symb}(\nu) \subseteq \mathrm{Symb}(\varphi_x(y), \Delta, \Lambda)$ we obtain $\mathrm{Symb}(\nu) \subseteq \mathrm{Symb}(\exists x \varphi, \Delta, \Lambda)$. And (iii), since $y$ is not free in the endsequent of (12), the following is a valid inference according to the antecedent $\exists$-rule:

$$\frac{\langle\, \varphi_x(y), \Delta, \nu \;\Rightarrow\; \Lambda \,\rangle}{\langle\, \exists x \varphi, \Delta, \nu \;\Rightarrow\; \Lambda \,\rangle.}$$

Thus the formula $\nu$, without any modification, satisfies the requirements. Reasoning in the other cases (principal formula in front of a semicolon or the succedent $\forall$-rule as the last step in $\mathcal{P}$) is completely analogous.

Assume now that the last step in $\mathcal{P}$ is an application of one of the rules (6). In addition, assume that the term $t$ is a variable, say $y$. We thus have a situation like this:

$$\frac{\langle\, \Gamma; \varphi_x(y), \Pi \;\Rightarrow\; \Delta; \Lambda \,\rangle}{\langle\, \Gamma; \forall x\varphi, \Pi \;\Rightarrow\; \Delta; \Lambda \,\rangle.} \tag{13}$$

It looks similar to (12), but now $y$ may occur free in any formula in the endsequent. Let again $\nu$ be a formula guaranteed by the induction hypothesis. If $y$ is free in $\exists x\varphi$ or in a formula in $\Pi$ or $\Lambda$, then $\mathrm{Symb}(\forall x\varphi, \Pi, \Lambda) = \mathrm{Symb}(\varphi_x(y), \Pi, \Lambda)$ and, no matter whether $y$ is free in it, the formula $\nu$ can be taken as the formula required for the endsequent. Otherwise we have $\mathrm{Symb}(\forall x\varphi, \Pi, \Lambda) = \mathrm{Symb}(\varphi_x(y), \Pi, \Lambda) - \{y\}$. Then

$$\frac{\langle\, \Gamma \;\Rightarrow\; \Delta, \nu \,\rangle}{\langle\, \Gamma \;\Rightarrow\; \Delta, \exists y\nu \,\rangle} \qquad \text{and} \qquad \frac{\dfrac{\langle\, \varphi_x(y), \Pi, \nu \;\Rightarrow\; \Lambda \,\rangle}{\langle\, \forall x\varphi, \Pi, \nu \;\Rightarrow\; \Lambda \,\rangle}}{\langle\, \forall x\varphi, \Pi, \exists y\nu \;\Rightarrow\; \Lambda \,\rangle}$$

are valid inferences because, in the second step in the right, the eigenvariable condition for $y$ is met. Thus $\mu = \exists y\nu$ is a formula required for the endsequent of (13).

The problematic case is when we have a complex term $t$ in the place of the variable $y$ in (13). Then $t$ may contain several symbols (function symbols and variables) that are not in $\mathrm{Symb}(\forall x\varphi, \Pi, \Lambda)$. These may occur in the formula $\nu$, but must not occur in the formula needed for the endsequent of (13). This case, while unresolved, is the main reason for writing this paper.

## 4 The presence or absence of the equality symbol

Let $L$ be the language $\{P, R, G\}$ where $P$ is a unary predicate, $R$ is a binary predicate and $G$ is a unary function symbol, and consider the sequent

$$\begin{aligned} \langle\, & R(G(G(z)), G(z)), \forall x \forall y(P(x) \;\&\; P(y) \to x = y)^{(\alpha)}, \\ & \qquad \forall x \forall y(R(x,y) \to \neg P(x) \;\&\; P(y))^{(\beta)} \;\Rightarrow\; \neg P(z) \,\rangle \end{aligned} \tag{14}$$

It is easy to verify that in logic with equality this sequent is logically valid:

> From $R(G(G(z)), G(z))$ we obtain $\neg P(G(G(z)))$ and $P(G(z))$ using the sentence $\beta$ in the antecedent of (14). Assume that $P(z)$. Then $P(G(z))$ together with $\alpha$ yield $z = G(z)$. From this we obtain $G(z) = G(G(z))$, and then from $P(G(z))$ we have $P(G(G(z)))$, which is is a contradiction.

Now consider the sequent

$$\begin{aligned} \langle\, & \exists u \exists v R(u,v), \forall x \forall y(P(x) \;\&\; P(y) \to x = y), \\ & \qquad \forall x \forall y(R(x,y) \to \neg P(x) \;\&\; P(y)) \Rightarrow \neg P(z) \,\rangle \end{aligned} \tag{15}$$

Since $G$ does not occur in it, it can be derived from (14) using the left rule in (7). However, it is straightforward to see that it is not logically valid. For this it is sufficient to pick a two-element structure $\mathcal{D}$ with a domain $D = \{a, b\}$ such that $R^{\mathcal{D}} = \{[a, b]\}$

and $P^{\mathcal{D}} = \{b\}$, and evaluate the variable $z$ by $b$. This example shows that the rules (7) are not sound with respect to the classical semantics for logic with equality. In the following theorem and in its proof we write $\underline{x}$ and $\underline{t}$ to denote an $n$-tuple.

**Theorem 1** *Let $\varphi$ be a formula, let $x_1, .., x_n$ be distinct variables and let $t_1, .., t_n$ be distinct terms such that every $t_i$ is substitutable for $x_i$ in $\varphi$. Furthermore, assume that if $t_i$ is a variable, then it has no free occurrences in $\Gamma \cup \Delta \cup \{\exists \underline{x} \varphi\}$, and if $t_i$ is a complex term, then its outermost symbol does not occur in $\Gamma \cup \Delta \cup \{\exists \underline{x} \varphi\}$. Then, in logic without equality, if $\langle\, \Gamma, \varphi_{\underline{x}}(\underline{t}) \;\Rightarrow\; \Delta \,\rangle$ is logically valid, then $\langle\, \Gamma, \exists \underline{x} \varphi \;\Rightarrow\; \Delta \,\rangle$ is logically valid, and if $\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi_{\underline{x}}(\underline{t}) \,\rangle$ is logically valid, then $\langle\, \Gamma \;\Rightarrow\; \Delta, \forall \underline{x} \varphi \,\rangle$ is logically valid.*

**Proof** Since the two claims are symmetric, it is sufficient to deal with the existential quantification. Let $L$ be the set of all function and predicate symbols in $\langle\, \Gamma, \exists \underline{x} \varphi \;\Rightarrow\; \Delta \,\rangle$. Let $G_1, .., G_m$ be the (distinct) function symbols that appear in $t_1, .., t_n$ as the outermost symbols, and let $y_1, .., y_k$ be those $t_i$ that are variables. The symbols $G_j$ are not in $L$. The terms $t_1, .., t_n$ may contain inner occurrences of further function symbols (the symbols $G_j$ included) and of variables (the variables $y_j$ included). Some of them may share the outermost symbol. We assume that $\langle\, \Gamma, \exists \underline{x} \varphi \;\Rightarrow\; \Delta \,\rangle$ is not logically valid and we aim to show that $\langle\, \Gamma, \varphi_{\underline{x}}(\underline{t}) \;\Rightarrow\; \Delta \,\rangle$ is not logically valid either. We thus start with a semantic counterexample for $\langle\, \Gamma, \exists \underline{x} \varphi \;\Rightarrow\; \Delta \,\rangle$. It consists of a structure $\mathcal{D}$ for $L$ and a valuation $e_0$ of variables in $\mathcal{D}$ such that $\mathcal{D} \models \psi[e_0]$ for every $\psi \in \Gamma \cup \{\exists \underline{x} \varphi\}$ and $\mathcal{D} \not\models \psi[e_0]$ for every $\psi \in \Delta$. Note that we use square brackets to enclose a valuation when writing the relation "satisfies" symbolically. Since $\mathcal{D} \models (\exists \underline{x} \varphi)[e_0]$, we can fix elements $a_1, .., a_n$ of the domain $D$ of $\mathcal{D}$ such that

$$\mathcal{D} \models \varphi[e_0(x_1/a_1, .., x_n/a_n)]. \tag{i}$$

Here $e_0(x_1/a_1, .., x_n/a_n)$ denotes the valuation that maps $x_1, .., x_n$ to $a_1, .., a_n$ and agrees with $e_0$ at all other variables. Let $U$ be the set of all terms in $L \cup \{G_1, .., G_m\}$. We put $M = D \times U$ and we fix an arbitrary $a_0 \in D$. The realization $F^{\mathcal{M}}$ of an $r$-ary function symbol $F \in L$, the realization $R^{\mathcal{M}}$ of an $r$-ary relation symbol $R \in L$, and the realizations $G_j^{\mathcal{M}}$ of the symbols $G_1, .., G_m$ are defined as follows:

$$F^{\mathcal{M}}([b_1, s_1], .., [b_r, s_r]) = [F^{\mathcal{D}}(b_1, .., b_r), F(s_1, .., s_r)], \tag{ii}$$

$$R^{\mathcal{M}}([b_1, s_1], .., [b_r, s_r]) \;\Leftrightarrow\; R^{\mathcal{D}}(b_1, .., b_r), \tag{iii}$$

$$G_j^{\mathcal{M}}([b_1, s_1], .., [b_r, s_r]) = \begin{cases} [a_i, G_j(s_1, .., s_r)] & \text{if } G_j(s_1, .., s_r) \text{ is } t_i \\ [a_0, G_j(s_1, .., s_r)] & \text{otherwise.} \end{cases} \tag{iv}$$

Since $t_1, .., t_n$ are pairwise different, a term $G_j(\underline{s})$ can equal at most one $t_i$, and so (iv) is a correct definition. The square brackets in (ii)–(iv) denote pairing. We suppose that this use can be easily distinguished from the situations where they enclose a valuation of variables (and the symbol $\models$ is involved). Let $g : M \to D$ and $h : M \to D$ be the left and right projections, i.e. the functions satisfying $g([b, s]) = b$ and $h([b, s]) = s$. Let $\mathcal{M}^-$ be the reduct of $\mathcal{M}$ to $L$, i.e. the structure obtained from $\mathcal{M}$ by omitting the realizations of $G_1, .., G_m$. Then it is clear from (ii) and (iii) that $g$ preserves all symbols in $L$. Thus $g$ is a homomorphism from $\mathcal{M}^-$ to $\mathcal{D}$. Note that in predicate logic without equality a homomorphism does not have to be one to one.

Consider a valuation $e$ in $\mathcal{M}$, a term $s$ and its value $s^{\mathcal{M}}[e]$ in $\mathcal{M}$ with respect to $e$. The function $g \circ e$ is a valuation in $\mathcal{D}$. If $s$ is a variable, then $s^{\mathcal{M}}[e]$ is $e(s)$ and the equality $g(e(s)) = (g \circ e)(s)$ can be written as $g(s^{\mathcal{M}}[e]) = s^{\mathcal{D}}[g \circ e]$. Using (ii), it is easy to prove that this equality holds for every term $s$ in $L$. From this and (iii) it follows that $\mathcal{M}^- \models \psi[e] \Leftrightarrow \mathcal{D} \models \psi[g \circ e]$ for every atomic formula $\psi$ in $L$. The fact that $g$ is onto and another induction show that the latter equivalence holds for every formula $\psi$ in $L$. Thus *$g$ preserves all formulas in $L$*. Since $\mathcal{M}^- \models \psi[e]$ is equivalent to $\mathcal{M} \models \psi[e]$ for $\psi$ in $L$, we have obtained

$$\mathcal{M} \models \psi[e] \Leftrightarrow \mathcal{D} \models \psi[g \circ e] \tag{v}$$

for each formula $\psi$ in $L$ and every valuation $e$ in $\mathcal{M}$. We now define a valuation $e_1$ in $\mathcal{M}$ as follows:

$$e_1(z) = \begin{cases} [a_i, z] & \text{if } z \text{ is } t_i \\ [e_0(z), z] & \text{otherwise.} \end{cases} \tag{vi}$$

The variables that equal some $t_i$ are $y_1, .., y_k$. Clearly, $g \circ e_1$ and $e_0$ agree at all other variables. Since $y_1, .., y_k$ are not free in $\Gamma \cup \Delta$ and $e_0$ satisfies in $\mathcal{D}$ all formulas in $\Gamma$ and none formula in $\Delta$, it follows from (v) that $\mathcal{M} \models \psi[e_1]$ for every $\psi \in \Gamma$ and $\mathcal{M} \not\models \psi[e_1]$ for every $\psi \in \Delta$. It remains to deal with the formula $\varphi_{\underline{x}}(\underline{t})$.

From (vi) and (ii) it is clear that $h(s^{\mathcal{M}}[e_1]) = s$ for every term $s$ in $L \cup \{G_1, .., G_m\}$. If $t_i$ has the form $G_j(s_1, .., s_r)$, then from (iv) we see that $g(t_i^{\mathcal{M}}[e_1]) = a_i$. If $t_i$ is $y_j$, then from (vi) we have $g(t_i^{\mathcal{M}}[e_1]) = a_i$ as well. Since $h(t_i^{\mathcal{M}}[e_1]) = t_i$, we have verified that $t_i^{\mathcal{M}}[e_1] = [a_i, t_i]$ for every $i \in \{1, .., n\}$. Then we have:

$$\begin{aligned} \mathcal{M} \models \varphi_{\underline{x}}(\underline{t})[e_1] &\Leftrightarrow \mathcal{M} \models \varphi[e_1(x_1/[a_1, t_1], .., x_n/[a_n, t_n])] \\ &\Leftrightarrow \mathcal{D} \models \varphi[g \circ e_1(x_1/[a_1, t_1], .., x_n/[a_n, t_n])], \end{aligned} \tag{vii}$$

where the first equivalence is an elementary fact about the truth value (w.r.t. a structure and an evaluation) of a formula obtained by substitution, and the second equivalence follows from (v). From (vi) we see that the valuations $g \circ e_1(x_1/[a_1, t_1], .., x_n/[a_n, t_n])$ and $e_0(x_1/a_1, .., x_n/a_n)$ agree at all variables $z$ that are different from all $x_1, .., x_n$ and all $y_1, .., y_k$. They also agree at $x_1, .., x_n$. The remaining variables are those $y_j$ that are not among $x_1, .., x_n$. Since $y_1, .., y_k$ are not free in $\exists \underline{x} \varphi$, those of them that are not among $x_1, .., x_n$ are not free in $\varphi$. Thus $g \circ e_1(x_1/[a_1, t_1], .., x_n/[a_n, t_n])$ and $e_0(x_1/a_1, .., x_n/a_n)$ agree at all variables that are free in $\varphi$. Then from (i) we have $\mathcal{D} \models \varphi[g \circ e_1(x_1/[a_1, t_1], .., x_n/[a_n, t_n])]$, and (vii) yields $\mathcal{M} \models \varphi_{\underline{x}}(\underline{t})[e_1]$. □

## 5 An example

We finish by an example on the use of Theorem 1. It is not difficult to verify that in predicate logic without equality, where there are no assumptions about the symbol $=$, the sequent (14) is not logically valid. However, adding $\forall x \forall y(x = y \rightarrow G(x) = G(y))^{(\gamma)}$ and $\forall x \forall y(x = y \rightarrow (P(x) \rightarrow P(y)))^{(\delta)}$ to its antecedent yields a logically valid sequent. Indeed, one can check that these two sentences are everything that is needed to make the informal proof in the beginning of Section 4 gap-free:

> Assume that $P(z)$. From $R(G(G(z)), G(z))$ and $\beta$ we have $\neg P(G(G(z)))$ and $P(G(z))$. Then $P(z)$ and $P(G(z))$ yield $z = G(z)$ using $\alpha$. From $\gamma$ we have $G(z) = G(G(z))$, and then $\delta$ yields $P(G(z)) \rightarrow P(G(G(z)))$. Since $P(G(z))$, we conclude that $P(G(G(z)))$, which is a contradiction.

This can be translated to a proof $\mathcal{P}$ of $\langle\, R(G(G(z)), G(z)), \gamma; \beta, \alpha, \delta \Rightarrow ; \neg P(z)\,\rangle$. Let the endsequent and thus the entire $\mathcal{P}$ be divided as indicated by the semicolons, let $\Gamma$ and $\Pi$ be $\{R(G(G(z)), G(z)), \gamma\}$ and $\{\beta, \alpha, \delta\}$, and put $\Delta = \emptyset$ and $\Lambda = \{\neg P(z)\}$. We have $\mathrm{Symb}(\Gamma, \Delta) = \{R, G, z, =\}$ and $\mathrm{Symb}(\Pi, \Lambda) = \{R, P, z, =\}$. We thus seek a formula $\mu$ satisfying $\mathrm{Symb}(\mu) \subseteq \{R, z, =\}$. The proof $\mathcal{P}$ contains no generalizations and in its construction we have a lot of freedom when choosing the order of instantiations. Assume that it ends by two unsubstitutions that yield the sentence $\beta$:

$$\cfrac{\cfrac{\langle\, \Gamma; R(G(G(z)), G(z)) \rightarrow \neg P(G(G(z))) \;\&\; P(G(z)), \alpha, \delta \Rightarrow ; \Lambda\,\rangle}{\langle\, \Gamma; \forall y(R(G(G(z)), y) \rightarrow \neg P(G(G(z))) \;\&\; P(y)), \alpha, \delta \Rightarrow ; \Lambda\,\rangle}}{\langle\, \Gamma; \forall x \forall y(R(x, y) \rightarrow \neg P(x) \;\&\; P(y)), \alpha, \delta \Rightarrow ; \Lambda\,\rangle.} \tag{16}$$

Writing down the entire proof $\mathcal{P}$ and revisiting Section 4, the reader can verify that the procedures described there yield the following formula $\nu$ for the upper sequent of (16):

$$R(G(G(z)), G(z)) \;\&\; (z = G(z) \rightarrow G(z) = G(G(z))).$$

Notice that the symbol $G$ occurs on both sides of semicolons in the upper sequent of (16) and thus it does not matter that it occurs in $\nu$. Let $\mu$ be

$$\exists u \exists v(R(u, v) \;\&\; (z = v \rightarrow v = u)).$$

Since $\langle\, \Gamma \Rightarrow \nu\,\rangle$ is logically valid, it is clear that $\langle\, \Gamma \Rightarrow \mu\,\rangle$ is logically valid. Also $\langle\, R(G(G(z)), G(z)) \rightarrow \neg P(G(G(z))) \;\&\; P(G(z)), \alpha, \delta, \nu \Rightarrow \Lambda\,\rangle$ is logically valid, and two instantiations applied to it yield $\langle\, \beta, \alpha, \delta, \nu \Rightarrow \Lambda\,\rangle$. The latter sequent is

$$\langle\, \Pi, R(G(G(z)), G(z)) \;\&\; (z = G(z) \rightarrow G(z) = G(G(z))) \Rightarrow \Lambda\,\rangle.$$

Now, as $G$ does not occur in $\Pi \cup \Lambda$, Theorem 1 is applicable and yields $\langle\, \Pi, \mu \Rightarrow \Lambda\,\rangle$. Thus the formula $\mu$ has the required properties: both $\langle\, \Gamma \Rightarrow \Delta, \mu\,\rangle$ and $\langle\, \Pi, \mu \Rightarrow \Lambda\,\rangle$ are logically valid and we have $\mathrm{Symb}(\mu) \subseteq (\Gamma \cup \Delta) \cap (\Pi \cup \Lambda)$.

## 6 Comments and conclusions

Let us again consider the situation described in the end of Section 3. Assume that an instantiation rule:

$$\frac{\langle\, \Gamma; \Pi, \theta_z(s) \Rightarrow \Delta; \Lambda\,\rangle}{\langle\, \Gamma; \Pi, \forall z \theta \Rightarrow \Delta; \Lambda\,\rangle} \tag{17}$$

is used in a cut-free proof $\mathcal{P}$ and that we have an interpolant $\mu$ of the upper sequent. Then $\mathrm{Symb}(\mu) \subseteq \mathrm{Symb}(\Gamma, \Delta) \cap \mathrm{Symb}(\Pi, \theta_z(s), \Lambda)$ and the sequents $\langle\, \Gamma \Rightarrow \Delta, \mu\,\rangle$

and $\langle\, \Pi, \theta_z(s), \mu \;\Rightarrow\; \Lambda \,\rangle$ are provable (logically valid). The term $s$ and thus also the formula $\mu$ can contain function symbols and free variables that do not occur (free) in $\mathrm{Symb}(\Pi, \forall z\theta, \Lambda)$. These symbols are *unwanted* because they must not occur in a possible interpolant of the bottom sequent in (17). If no occurrences of variables in the scope of unwanted function symbols are bound, we can write $\mu$ as $\varphi_{\underline{x}}(t_1, .., t_n)$ where the terms $t_1, .., t_n$ are as described in Theorem 1. Then $\langle\, \Gamma \;\Rightarrow\; \Delta, \exists\underline{x}\varphi \,\rangle$ is provable from $\langle\, \Gamma \;\Rightarrow\; \Delta, \varphi_{\underline{x}}(\underline{t}) \,\rangle$ via instantiations, and the provability of $\langle\, \Pi, \forall z\theta, \exists\underline{x}\varphi \;\Rightarrow\; \Lambda \,\rangle$ follows from the provability of $\langle\, \Pi, \forall z\theta, \varphi_{\underline{x}}(\underline{t}) \;\Rightarrow\; \Lambda \,\rangle$ using Theorem 1. Then $\exists\underline{x}\varphi$ is an interpolant of the bottom sequent in (17). However, a problem is that if (17) is not the last inference in the proof $\mathcal{P}$, then symbols that are unwanted at this stage may occur in the scope of function symbols that become unwanted at some later stage. Then getting rid of unwanted symbols (that is, generalizing the terms $t_1, .., t_n$) at this stage introduces bound occurrences of variables, and the just described procedure cannot be simply repeated at later stages.

This explains that Theorem 1 is probably not sufficient to prove the general interpolation theorem for classical predicate logic without equality but with function symbols. It can only solve some cases, as the example in Section 5 suggests.

## References


[Bíl07] M. Bílková. Uniform interpolation and propositional quantifiers in modal logics. *Studia Logica*, 85(1):1–31, 2007.

[Bus98] S. R. Buss. An introduction to proof theory. In S. R. Buss, ed., *Handbook of Proof Theory*, no. 137 in Studies in Logic and the Foundations of Mathematics, chap. I, pp. 1–78. Elsevier, 1998.

[Cra57a] W. Craig. Linear reasoning. A new form of the Herbrand-Gentzen theorem. *J. Symb. Logic*, 22(3):250–268, 1957.

[Cra57b] W. Craig. Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. *J. Symb. Logic*, 22(3):269–285, 1957.

[GR08] R. Goré and R. Ramanayake. Valentini's cut-elimination for provability logic resolved. In C. Areces and R. Goldblatt, eds., *Advances in Modal Logic 7 (AiML'08 Nancy)*, pp. 67–86. College Publications, London, 2008.

[Hen63] L. Henkin. An extension of the Craig-Lyndon interpolation theorem. *J. Symb. Logic*, 28:201–216, 1963.

[Lyn59a] R. C. Lyndon. An interpolation theorem in the predicate calculus. *Pacific J. Math.*, 9(1):129–142, 1959.

[Lyn59b] R. C. Lyndon. Properties preserved under homomorphism. *Pacific J. Math.*, 9(1):143–154, 1959.

[Min02] G. Mints. *A Short Introduction to Intuitionistic Logic*. The University Series in Mathematics. Kluwer, 2002.

[MOU13] G. Mints, G. Olkhovikov, and A. Urquhart. Failure of interpolation in constant domain intuitionistic logic. *J. Symb. Logic*, 78(3):937–950, 2013.

[SV82] G. Sambin and S. Valentini. The modal logic of provability: The sequential approach. *J. Phil. Logic*, 11:311–342, 1982.

[Sol76] R. M. Solovay. Provability interpretations of modal logic. *Israel J. Math.*, 25:287–304, 1976.

[Tak75] G. Takeuti. *Proof Theory*. North-Holland, 1975.

ACTA UNIVERSITATIS CAROLINAE
**PHILOSOPHICA ET HISTORICA 1/2022**

Edited by Šárka Stejskalová
Cover and layout by Kateřina Řezáčová
Published by Charles University
Karolinum Press, Ovocný trh 560/5, 116 36 Prague 1
www.karolinum.cz, journals@karolinum.cz
Prague 2025
Typeset by Vítězslav Švejdar
Printed by Karolinum Press
ISSN 0567-8293 (Print)
ISSN 2464-7055 (Online)
MK ČR E 18598

Distributed by Faculty of Arts, Charles University,
2 Jan Palach Sq., 116 38 Prague 1, Czech Republic
(books@ff.cuni.cz)